UNIVERSIDADE FEDERAL DE LAVRAS

NATHAN BASTOS XAVIER

ASPECTS OF NON-HERMITIAN QED

LAVRAS – MG
2024

NATHAN BASTOS XAVIER

ASPECTS OF NON-HERMITIAN QED

Dissertação apresentada à Universidade Federal de Lavras, como parte das exigências do Programa de Pós-Graduação em Física, área de concentração em Física, para a obtenção do título de Mestre.

Prof. Dr. Rodrigo Santos Bufalo
Orientador

LAVRAS – MG
2024

Ficha catalográfica elaborada pelo Sistema de Geração de Ficha Catalográfica da Biblioteca Universitária da UFLA, com dados informados pelo próprio autor.

Xavier, Nathan Bastos
Aspects of non-Hermitian QED / Nathan Bastos Xavier. - 2024.
96 p. : il.

Orientador: Prof. Dr. Rodrigo Santos Bufalo.
Dissertação (mestrado acadêmico) - Universidade Federal de Lavras, 2024.
Bibliografia.

1. Extensões da QED. 2. Simetria $\mathcal{PT}$. 3. Limite não relativístico. 4. Modelo quiral de Schwinger. 5. Anomalia quiral. I. Bufalo, Rodrigo Santos. II. Título.

NATHAN BASTOS XAVIER

ASPECTS OF NON-HERMITIAN QED

ASPECTOS DA QED NÃO HERMITIANA

Dissertação apresentada à Universidade Federal de Lavras, como parte das exigências do Programa de Pós-Graduação em Física, área de concentração em Física, para a obtenção do título de Mestre.

APROVADA em 29 de Julho de 2024.

Prof. Dr. Luiz Cleber Tavares de Brito	UFLA
Prof. Dr. Alesandro Ferreira dos Santos	UFMT
Prof. Dr. Manoel Messias Ferreira Junior	UFMA

Documento assinado digitalmente
gov.br
RODRIGO SANTOS BUFALO
Data: 26/09/2024 14:10:46-0300
Verifique em https://validar.iti.gov.br

Prof. Dr. Rodrigo Santos Bufalo
Orientador

LAVRAS – MG
2024

To my family and my partner.

ACKNOWLEDGMENTS

I would like to thank my family for their support, especially my parents Suzamara and Telmo. I would also like to thank my partner Juliana for the shared moments of happiness. I would like to thank my advisor Rodrigo for his tutoring. I would like to thank my postgraduate colleagues for their company during my free time. And, finally, I acknowledge the support from Fundação de Amparo à Pesquisa do Estado de Minas Gerais (FAPEMIG Agreement for RD&I for the Granting of Scholarship Quotas No. 5.02/2022).

O verdadeiro amor exibe simetria de inversão temporal: há de ser no futuro tal e qual fora no passado.

Xavier, N. B.

ABSTRACT

With the increasing research interest in the $\mathcal{PT}$-symmetric non-Hermitian physics it is natural to investigate the Quantum Electrodynamics (QED) exhibiting such properties. Therefore, it is replicated and examined for this QED extension some well-known results of usual QED, including: (i) The non-relativistic limit of a scattering electron by a classical electromagnetic field, which results into the Pauli-Schrödinger Hamiltonian; (ii) The vacuum polarization tensor for the Chiral Schwinger Model, which is this QED extension in (1+1) dimensions; and (iii) The chiral anomaly for the Chiral Schwinger Model. New couplings in the Pauli-Schrödinger Hamiltonian for this QED extension were observed; and some limits were established on the model parameters by using the toroidal moment of the neutrino, the dipole eletric moment of the electron, the anomalous magnetic moments of electron, and the ground-energy state of the hydrogen atom. Concerning to the Chiral Schiwnger Model, it was observed some kind of mass of the photon, but to better comprehension about the dynamical mass generation in this model it is necessary to bosonizate it.

Keywords: QED extensions; $\mathcal{PT}$-symmetry; non-relativistic limit; chiral Schwinger model; chiral anomaly.

RESUMO

Com o crescente interesse de pesquisa na física não Hermitiana com simetria $\mathcal{PT}$, é natural investigar a Eletrodinâmica Quântica (QED) exibindo tais características. Portanto, replica-se e examina-se, para essa extensão da QED, alguns dos resultados bem conhecidos da QED usual, incluindo: (i) o limite não relativístico de um espalhamento de um elétron por um campo eletromagnético clássico, cujo produto é o Hamiltoniano de Pauli-Schrödinger; (ii) o tensor de polarização do vácuo para o Modelo Quiral de Schwinger, que é essa extensão da QED em (1+1) dimensões; e (iii) a anomalia quiral para o Modelo Quiral de Schwinger. Foram observados novos acoplamentos no Hamiltoniano de Pauli-Schrödinger para essa extensão da QED; e estabeleceu-se alguns limites nos parâmetros do modelo ao usar o momento toroidal do neutrino, o momento de dipolo do elétron, o momento magnético anômalo do elétron e a energia do estado fundamental do átomo de hidrogênio. No que diz respeito ao Modelo Quiral de Schwinger, obversou-se um tipo de massa do fóton, mas para uma melhor compreensão acerca da geração dinâmica de massa nesse modelo é necessário bosonizá-lo.

Palavras-chave: extensão da QED; simetria $\mathcal{PT}$; limite não relativístico; modelo quiral de Schwinger; anomalia quiral.

IMPACT INDICATORS

As it is a theoretical work in Physics, there are no social, economic or technological impacts. However, this work gives impact to the culture of Physics, given that the objective of investigating the non-relativistic limit of the scattering of a non-Hermitian fermion had been achieved, in which both new couplings (electric and toroidal dipole) were observed, previously not seen in the usual Hermitian model, as well as corrections to the couplings already known in the usual Hermitian model (electric, angular momentum and spin). Furthermore, using the experimental values of the electron electric dipole moment and its anomalous magnetic moment, it was possible to restrict the parametric region accessible to the axial parameters of the model; the same was done using the error associated with the fundamental energy level of the Hydrogen atom. Such results contribute to new advances in research on non-Hermitian models and can stimulate future results that culminate in a better understanding of physical nature.

INDICATORES DE IMPACTO

Por tratar-se de um trabalho teórico em Física, não há impactos de ordem social, econômica e tecnológica. No entanto, esta obra confere impacto à cultura da Física, dado que o objetivo de investigar o limite não relativístico do espalhamento de um férmion não Hermitiano fora alcançado, no qual se observou tanto novos acoplamentos (dipolo elétrico e toroidal), antes não vistos no modelo Hermitiano usual, quanto correções aos acoplamentos já conhecido no modelo Hermitiano usual (elétrico, momento angular e spin). Além do mais, utilizando os valores experimentais do momento de dipolo elétrico elétron e de seu momento magnético anômalo, foi possível restringir a região paramétrica acessível aos parâmetros axiais fo modelo; o mesmo foi feito usando o erro associado ao nível de energia fundamental do átomo de Hidrogênio. Tais resultados contribuem para os novos avanços na pesquisa de modelos não Hermitianos e podem estimular futuros resultados que culminaram numa melhor compreensão da natureza física.

LIST OF FIGURES

LIST OF TABLES

CONTENTS

1 INTRODUCTION

The Standard Model (SM), besides being supported by a solid theoretical basis — local relativistic quantum field theory, gauge symmetry and renormalizability (WILCZEK, 2004, p. 4) —, is formidable by describe both action and interaction of these fundamental elements of matter: the particles.

It is reinforced that, as highlighted by Goldberg (2017, p. xiv), the SM is not just a list of fundamental interaction; exordial is such that one can consider it as a theory of the symmetry of vacuum itself, with which quantum fields, whose excitations result in the aforementioned particles,[1] interact and occupy it[2]

Concretely, the SM augurs experimental results with great accuracy, given the anomalous magnetic moment of electron (AOYAMA; KINOSHITA; NIO, 2019) and the observation of the Higgs boson (AAD et al., 2012; CHATRCHYAN et al., 2012), which is the manifestation of excitation of the required gauge field by the mass generation mechanism.

However, this theory has certain limitation, the most notable being the lack of a compendium of gravitation interaction[3] and, according to Goldberg (2017, p. 253-255), the relatively high number of 19 free parameters and their enigmatic values obtained from experimental data.[4] Furthermore, it is unable to provide explanations for certain evident phenomena, such as neutrino oscillations (BILENKY, 2018, chap. 11; SUEKANE, 2015, chap. 5), and, on the other hand, it is not fully favored by measurable observables in nature, such as the anomalous magnetic moment of the muon (ABI et al., 2021).

[1] There is also a even more abstract definition of particle: It is that which is transformed under unitary and irreducible representations of the Poincarè group, cf. Schwartz (2014, p. 110).

[2] The idea of correspondence between particles and quantum fields only really exists in free theory. In a theory with interaction, this correspondence can only be established asymptotically in the calculation of Z matrix (see., e.g., the LSZ formalism).

[3] Only on the Planck scale. At usual physics scales, we have a perturbative version with predictive power in the context of an effective QFT. See, e.g., Donoghue (1994).

[4] Such values, together with the dimensinality of spacetime, cf. Dyson (1971), Tegmark (1997) and Rees (2003), raise questions about the adjustability of the universe to the existence of a type of life capable of understanding it, resulting in the so-called anthropic principle: The most probable values of physics constants are those conducive to generating intelligent observers capable of observing the universe, cf. Goldberg (2017). See Weinberg (1987) for a important result given that it occurred before the discovery of the non-zero value of the cosmological constant.

Briefly, there is, therefore, a good theory that sublimely explains certain phenomena, although it is flawed and limited in other aspects.[5]

Therefore, in view of the theoretical solidity in which the SM is established, and also of its success when compared with experimental data, it must be agreed that it is prudent not to give up on this theory due to its flaws, but look for extensions that, in addition to the successes *a posteriori*, can add, filling in the gaps and offering answers to fundamental questions in physics.

There are consolidated methods for searching extensions of the SM. One that stands out, like electroweak unification, is the search for unified theories. However, these models face, according to Goldberg (2017, sec. 12.2), significant difficulties, such as the large number of interaction mediators. Another approach is the creation of supersymmetric models.

According to Millington (2022, p. 2), the aforementioned methods are consistent with the characterization of the insertion of new degrees of freedom. There is, however, another characterization: Relaxing basic assumptions of the SM, such as locality, Lorentz invariance, number of spatial dimensions and Hermiticity.

In the research developed here, the hypothesis to be relaxed is the Hermitian quality of the Action or, equivalently, of the model-originating Hamiltonian.

However, the Hermiticity is one of the main characteristics of Quantum Mechanics. On the one hand, when the Hermiticity is a symmetry of the Hamiltonian, it guarantees both the reality of energy spectrum of a quantum state and the conservation of probability of isolated quantum systems, as said by Ashida, Gong, and Ueda (2020, p. 251); on the other hand, it is used to construct a positive-defined inner product, which is essential to the probabilistic interpretation of Quantum Mechanics. There is, therefore, a major paradigm shift in allowing this hypothesis to be relaxed, causing a reassessment of the reality of energy[6] and the inner product.

Precisely, although, how the Hermiticity as a symmetry of system described by a Hamiltonian $\hat{H}$ is defined? It establishes the condition that this Hamiltonian is invariant under the

[5] See that the SM manage to be in agreement with the vast majority of data available since Experimental High Energy Physics began. The limitations occur at a very fundamental level. In fact, perhaps the resolution of these issues involves a paradigm shift, with something very different from a QFT.

[6] However, it is interesting to highlight that, in Physics, the energy belonging to the body of complexes is not a chimera, although terrifying to the unwary as it would be immeasurable. For example, Peierls (1959, p. 16) already highlighted that resonance peaks in scattering are linked to complex energy eigenvalues, of which the imaginary part designates the thickness of the resonance.

Hermitian conjugation $\dagger$: $\hat{H}^\dagger = \hat{H}$, where $\dagger = \intercal \cdot *$ is the Hermitian operation, which for matrix representation is the application of complex conjugation followed by the transposition operation.

Now introduced the Hermiticity axiom one can ask the following: Why do such axioms show themselves in a physical language with the exception of Hermiticity, which appears under a strictly mathematical language?

It would then be appropriate to allow the relax of the Hermiticity as long as it is replaced by a key physical concept, generalizing it. And according to Bender (2019, p. x) there is a symmetry that can supersede the Hermitian symmetry: $\mathcal{PT}$ symmetry, which is the combination of discrete parity transformations $\mathcal{P}$ and temporal inversion $\mathcal{T}$.

However, how does the symmetry under such transformations shows itself in a physical language? It turns out that both discrete transformations $\mathcal{P}$ and $\mathcal{T}$ are elements of the Lorentz group, which is the structural group of spacetime in the absence of gravity. It can then be said that the symmetry $\mathcal{PT}$ condition is demanded by the physical spacetime.

It is worth mentioning that the appeal to use the $\mathcal{PT}$ symmetry instead of Hermiticity when the latter is violated is not only due to the physical argument. Indeed, this symmetry is used to construct a new inner product satisfying properties as positive-defined norm and orthogonality of energy eigenstates, which are no longer satisfied by the Hermitian inner product. This construction will be illustrated in the section 2.1 for the two-level non-Hermitian $\mathcal{PT}$-symmetric quantum system.

There was already works on non-Hermitian quantum mechanics (HATANO; NELSON, 1996, 1997; NELSON; SHNERB, 1998) and non-Hermitian field theory (BENDER; MILTON, 1997), although without giving the deserved significance to the $\mathcal{PT}$ symmetry. Those who brought to light this new class of symmetric $\mathcal{PT}$ Hamiltonians were Bender and Boettcher (1998). By replacing the Hermiticity condition with $\mathcal{PT}$ symmetry, these authors obtained a new infinite class of non-Hermitian and symmetric $\mathcal{PT}$ Hamiltonians with a real energy spectrum, having transition points from which energy begins to exhibit complex values.

Subsequently, Mostafazadeh (2002a,b,c) gave mathematical rigor to the curious result mentioned above, generalizing the concept of Hermiticity by introducing the definition of pseudo-Hermiticity of the Hamiltonian. Thus, Hermitian or $\mathcal{PT}$ symmetric Hamiltonians that admit a complete and biorthonormal set of eigenvectors are subclasses of pseudo-Hermitian Hamiltonians.

Moreover, it is worth highlighting the physical importance of $\mathcal{PT}$ symmetry, which places it as a good threshold for the generalization of Hermiticity. A direct example lies in the analysis of an open quantum system that is in contact with an external environment, suffering a loss or gain of energy. If the system exhibits $\mathcal{PT}$ symmetry, the loss and gain are, according Bender, Berntson, et al. (2013, p. 3), precisely balanced, and, therefore, behaving effectively as if it is a closed system, although it is not in fact; if the symmetry is broken, it becomes an open system. Therefore, a system exhibiting this symmetry allows the transition between different types of physical systems.

The application of non-Hermitian models can be exemplified by the study of dissipative quantum dynamics using the stochastic Schrödinger equation (LIN et al., 2022). Those that exhibit $\mathcal{PT}$ symmetry, in turn, are used in photonics (KRISHNAMOORTHY et al., 2023, p. 4435), photonic topology (MIDYA; ZHAO; FENG, 2018), nanophotonic systems (KRASNOK; ALÙ, 2022, p . 13-14) and many-body quantum systems (ASHIDA; FURUKAWA; UEDA, 2017; MATSUMOTO et al., 2020), quantum computing (BENDER; BRODY, et al., 2007), and quantum information theory (JU et al., 2019).

Another use of non-Hermitian models, especially those with $\mathcal{PT}$ symmetry, is in quantum field theory (ALEXANDRE; BENDER, 2015; ALEXANDRE; MILLINGTON; SEYNAEVE, 2017; ALEXANDRE; ELLIS; MILLINGTON; SEYNAEVE, 2018; ALEXANDRE; ELLIS; MILLINGTON, 2020). Furthermore, it was noted, in this context, the usefulness of such models in describing neutrinos (ALEXANDRE; BENDER; MILLINGTON, 2015, 2017; OHLSSON, 2016; OHLSSON; ZHOU, 2020).

An interesting remark is the analogy between the finding made by Makris et al. (2008) in the field of optical networks with $\mathcal{PT}$ symmetry with some remarkable characteristic of neutrinos: its well-defined chirality. When studying the dynamics of beams in optical networks with $\mathcal{PT}$ symmetry, the aforementioned authors noticed that light distinguishes the spatial directions, left and right. Not only that, this turns out to be a general property of pseudo-Hermitian optical systems, as are those with $\mathcal{PT}$ symmetry. Since an analogy is made between spatial directions and the left- and right-hand chiralities of an elementary particle, this result alludes to neutrinos, since they must be treated under a model that can differentiate their chiralities.

Summarize, was made a short introduction to the Standard Particle Model. Some of its main problems was highlighted, which justified the searching for extensions of this model, where the method to be carried out in this work was delimited, namely the relaxation of Hermiticity in exchange for from the symmetry $\mathcal{PT}$.

Thus, in chapter 2, we will dedicate ourselves to presenting a theoretical framework that illustrates the potential of this approach.

In section 2.1, the simplest example of a non-Hermitian model with $\mathcal{PT}$ symmetry will be presented: the two-level system. It will be possible, with it, to discuss common situations of this type of system. Among such situations are the following: the reality of the energy spectrum depends on a model parameter and the inner product must be redefined so that it exhibits both the condition of orthonormality of the energy eigenstates and definite positivity.

In section 2.2, the non-Hermitian and $\mathcal{PT}$ symmetric model of a free fermionic field will be presented, where the different treatment that the different chiralities of this field receive in this model will be discussed.

In chapter 3, we will present the main result of this dissertation: the non-relativistic limit of a non-Hermitian $\mathcal{PT}$-symmetric fermion scattering off by a classical electromagnetic field, where we will looking for new couplings presents into the Pauli-Schrödinger Hamiltonian.

In chapter 4, we discussed some anomalies in the Schwinger chiral model with non-unitary couplings, including the photon mass in section 3.2, and the chiral anomaly in section 3.3.

Finally, in chapter 5 we give our final remarks and a possible research continuity by elaborating some outstanding questions.

2 THEORETICAL REFERENCE

This chapter is dedicated to present the principal results concerning to two non-Hermitian models already studied in the literature.

The first, presented in section 2.1, is the simplest non-Hermitian $\mathcal{PT}$-symmetric model: The two-level non-Hermitian $\mathcal{PT}$-symmetric quantum system. It is a good example to show how the symmetry $\mathcal{PT}$ plays a main role in several physical aspects of the model.

In this model, this symmetry distinguishes two regions in the spectrum of the Hamiltonian, namely, regions of unbroken and broken $\mathcal{PT}$ symmetry. The former, on the one hand, concerns the region where the energy eigenvalues are real and their associated eigenvectors are also eigenvectors of the $\mathcal{PT}$ transformation. The latter, on the other hand, concerns the region where the energy eigenvalues are not real, but complex and their associated eigenvector are not mutually eigenvectors of the $\mathcal{PT}$ transformation.

Furthermore, it will be shown that the usual Hermitian inner product does not allow the eigenstates of this non-Hermitian Hamiltonian to be orthogonal to each other, and therefore it is necessary to reformulate the inner product in order to allow this orthogonality, and this is done using the $\mathcal{PT}$ symmetry of the Hamiltonian. However, the new inner product is meaningful only for the region of unbroken $\mathcal{PT}$ symmetry.

The second model, presented in section 2.2, is the non-Hermitian $\mathcal{PT}$-symmetric free fermion model.

It will be shown that the parameter that confers the non-Hermiticy exhibits a mass behavior and is responsible for: (i) Establishing the region in which the energy is real; (ii) Demarcating the points at which the model becomes non-massive; and (iii) Recognizing regions in which a chirality, left-hand or right-hand, is predominant.

Finally, it is worth mentioning that this non-Hermitian fermionic model establishes the background to define the non-Hermitian and $\mathcal{PT}$-symmetric QED that will be treated in chapter 3.

2.1 Two-level non-Hermitian $\mathcal{PT}$-symmetric quantum system

Let a quantum system be described by a Hamiltonian. The following observations can then be made: (i) Given that the energy is an observable and the ground state must be stable, the energy spectrum is real and limited below; (ii) Given that the inner product is associated with the probability of states, it must be positive-defined; and (iii) The Hamiltonian operator must generate a unitary temporal evolution so that there is conservation of probability (BENDER; BERNTSON, et al., 2013, p. 83).

Such conditions are satisfied for real and symmetric Hamiltonians or complex and Hermitian operators; both are somewhat restrictive requirements, although the later is lesser restrictive than the first. A requirement that is even less restrictive than Hermiticity and that does not violates the three conditions mentioned above only those exhibiting $\mathcal{PT}$ symmetry, where non-hermiticity is allowed.

Bender (2019, sec. 1.1) give an excellent introduction to the physical implication of non-Hermitian Hamiltonians that exhibit $\mathcal{PT}$ symmetry, which, basically, are complex extensions of those real Hermitians.

The main aspect that distinguishes a non-Hermitian model with $\mathcal{PT}$ symmetry is its usefulness to describe both open and closed systems. The former has contact with the environment while the latter does not. Hermitian models is restricted to describe only systems of closed type.

Obviously, open systems are closer to reality than closed ones. Indeed, as they do not occur in isolated locations, real physical processes are subject to external influences, which characterize an open system. Describing them, however, is not a simple task.

An interesting system that emulates open system is that of gain-loss. Bender (2019, sec. 1.4), for example, present a classical system of coupled harmonic oscillators that exhibits $\mathcal{PT}$ symmetry: (i) For weak couplings, the symmetry is broken and the solutions of the oscillators' equations of motion have a well-defined oscillation frequency and grow exponentially unlimitedly; (ii) For strong couplings, on the other hand, for which symmetry is also broken, the solutions of the equations of motion grow so quickly that it does not even allow for there to be a defined oscillation frequency; (iii) For couplings overlapping the previous ones, the phenomenon known as Rabi oscillations occurs, which is similar to the phenomenon of acoustic beating [cf. Bender (2019, p. 16, Fig. 1.6)].

Here we will present a simple gain-loss model: The two-level non-Hermitian $\mathcal{PT}$-symmetric quantum system (BENDER; BERNTSON, et al., 2013), and we aimed in this section to provide a detailed and didactic treatment of this system, from which general characteristics of this kind of model will be elucidated.

This system is given by the following Hamiltonian (BENDER, 2019, sec. 1.2):

$$\hat{H} = \begin{pmatrix} a + ib & g \\ g & a - ib \end{pmatrix}, \tag{2.1.1}$$

where $a, b, g \in \mathbb{R}$.

The parameter g is responsible to couple the two subsystems composing the Hamiltonian, and we will see that it plays a fundamental role to establish the reality of the energy spectrum. The complex numbers in the main diagonal of $\hat{H}$ given in equation (2.1.1) meaning that separately, i.e, without coupling ($g = 0$), the two subsystems composing this Hamiltonian would have complex values of energy. So what we have, in short, is a system composed of two coupled open subsystems.

It is clear that the $\hat{H}$ is not Hermitian:

$$\hat{H}^{\dagger} = \begin{pmatrix} a - ib & g \\ g & a + ib \end{pmatrix} \neq \begin{pmatrix} a + ib & g \\ g & a - ib \end{pmatrix} = \hat{H}, \tag{2.1.2}$$

although it exhibits $\mathcal{PT}$ symmetry:

$$\mathcal{PT} : \hat{H} \rightarrow \hat{H}' = \hat{U}_{\mathcal{PT}} \hat{H} \hat{U}_{\mathcal{PT}}^{-1} = \hat{H}, \tag{2.1.3}$$

where $\hat{U}_{\mathcal{PT}}$ is given by

$$\hat{U}_{\mathcal{PT}} = \hat{U}_{\mathcal{P}} \mathcal{K} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{K}, \tag{2.1.4}$$

in which $\mathcal{K}$ stands for the complex conjugation operation, which represents the operator of time inversion transformation $\mathcal{T}$, and $\hat{U}_{\mathcal{P}}$ is the operator of the parity transformation $\mathcal{P}$.

If one solves the secular polynomial given by $\hat{H}$ the energy spectrum is found

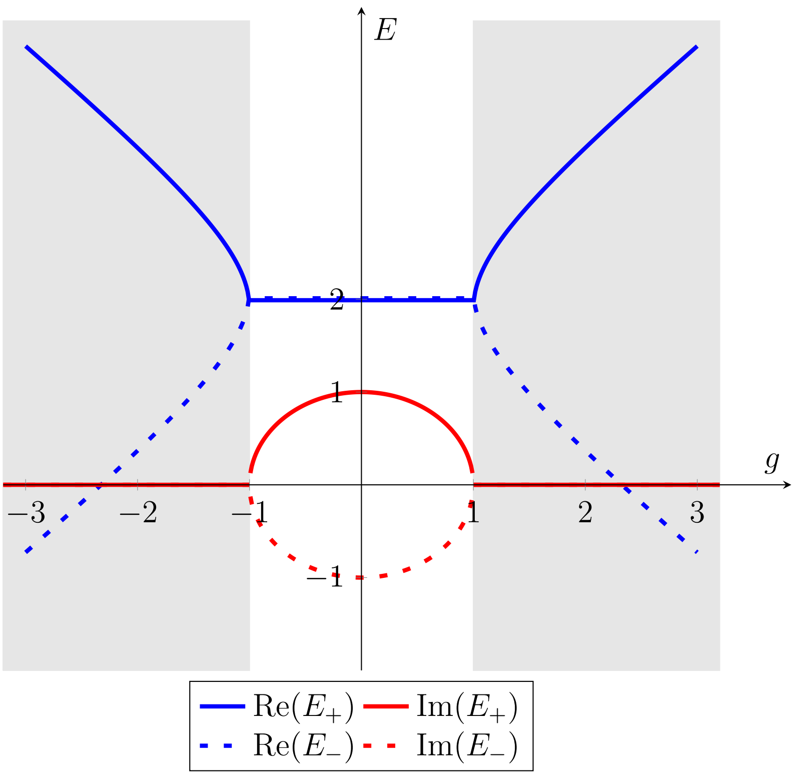
$$E_{\pm} = a \pm \sqrt{g^2 - b^2}. \tag{2.1.5}$$

There are some interesting information which can be learned from the energy eigenvalues given in equation (2.1.5).

Firstly, it can be seen that, for weak couplings, where $g^2 < b^2$, the square root becomes a purely imaginary number and, consequently, the energy becomes a complex value; this parametric region is known as a region of broken $\mathcal{PT}$ symmetry. On the other hand, for strong couplings, where $g^2 > b^2$, the energy is a real giving that the square root now assumes only real numbers; this region is known as a region of unbroken $\mathcal{PT}$ symmetry. Finally, one can notice the merging of eigenvalues under the condition that $g = \pm b$, so that $E_+ = E_-$; the points where this occurs are called exceptional points, which indicate the transitions between broken and unbroken phases of the $\mathcal{PT}$ symmetry.

The figure 2.1 illustrates the energy spectrum given in equation (2.1.5), where the energy is plot as a function of the coupling parameter g for the values $a = 2$ e $b = 1$. In this figure the regions where $\mathcal{PT}$ symmetry is broken and unbroken are distinguishable, and it is easy to see the points where phase transitions occur.

Figure 2.1 – Energy spectrum of the Hamiltonian $\hat{H}$ as a function of the coupling parameter g. The values $a = 2$ and $b = 1$ were chosen. The solid curves in blue and red indicate, respectively, the real and imaginary parts of the energy E_+; and the dashed curves in blue and red indicate, respectively, the real and imaginary parts of the energy E_-. In the region $g^2 > b^2$, shaded in gray, the $\mathcal{PT}$ symmetry is unbroken, i.e., the energies are real; in the region $g^2 < b^2$, this symmetry is broken and the energies are complex; finally, at the points $g = \pm 1$, $\mathcal{PT}$ symmetry phase transition occur.



Source: Authors (2024), based on Bender (2019, p. 10, Figure 1.4).

Now we present the eigenstates of $\hat{H}$ for the region of unbroken $\mathcal{PT}$ symmetry, i.e., for $g^2 > b^2$:

$$|\psi_+\rangle = N \begin{pmatrix} \sqrt{g^2 - b^2} + ib \\ g \end{pmatrix}, \quad |\psi_-\rangle = N \begin{pmatrix} -g \\ \sqrt{g^2 - b^2} + ib \end{pmatrix}, \quad N \in \mathbb{R}, \tag{2.1.6}$$

where $|\psi_\pm\rangle$ are associated, respectively, to the eigenvalues $E_\pm$; N is a normalization constant where its restriction to real numbers does not affect the generality of the solution, since the neglected complex phase does not interfere with quantum observables.

The usual hermitian normalization of the eigenstates $|\psi_\pm\rangle$ gives the following result for the normalization constant:

$$N = \frac{1}{\sqrt{2}|g|}, \tag{2.1.7}$$

so that $\langle\psi_\pm|\psi_\pm\rangle = 1$.

The orthogonality through the usual Hermitian inner product, on the other hand, yields

$$\langle\psi_+|\psi_-\rangle = \langle\psi_-|\psi_+\rangle^* = \frac{ib}{g}, \tag{2.1.8}$$

which is non-null.

Therefore, a natural question arises: Would it then be possible to restrict the system parameters so that orthogonality was satisfied?

Well, by observing the equation (2.1.8) it can be seen that there is two ways to nullify the inner product: (i) Restrict the value of b to zero, $b = 0$; (ii) Making the coupling g infinitely big, $g \to \infty$. The latter way, on the one hand, has a non-physical meaning, as energy takes on an infinite value. The first way, on the other hand, reduces the Hamiltonian $\hat{H}$ to the Hermitian class, as b is the parameter which gives the complex deformation for the Hamiltonian.

In order to avoid non-physical meaning and also reducing the model described by the non-Hermitian Hamiltonian $\hat{H}$ to the class of Hermitian models, one must have g finite and $b \neq 0$, and consequently the Hermitian inner product does not satisfies orthogonality for the eigenstates of the unbroken region of $\mathcal{PT}$ symmetry.

Now we present the eigenstates for the region of broken $\mathcal{PT}$, i.e., for $g^2 < b^2$:

$$|\phi_+\rangle = M \begin{pmatrix} i\left(b + \sqrt{b^2 - g^2}\right) \\ g \end{pmatrix}, \quad |\phi_-\rangle = M \begin{pmatrix} -g \\ i\left(b + \sqrt{b^2 - g^2}\right) \end{pmatrix}, \quad M \in \mathbb{R}, \tag{2.1.9}$$

where $|\phi_+\rangle$ e $|\phi_-\rangle$ are related, respectively, to E_+ e E_-; M is a normalization constant with the same characteristics as N.

The normalization via usual Hermitian inner product gives

$$M = \frac{1}{2b(b + \sqrt{b^2 - g^2})}, \tag{2.1.10}$$

so that $\langle\phi_\pm|\phi_\pm\rangle = 1$.

On the other hand, the orthogonalization through usual Hermitian inner product yields

$$\langle \phi_+ | \phi_- \rangle = \langle \phi_- | \phi_+ \rangle^* = \frac{ig}{b}. \tag{2.1.11}$$

Giving that $b = 0$ is a non-regular point of the previous result, we conclude that the orthogonality is not satisfied in the Hermitian limit. However, in the absence of coupling, for which $g = 0$, orthogonality is satisfied.

So the regions of unbroken and broken $\mathcal{PT}$ symmetry reserve another dissimilarity beyond the reality of energy: The orthogonality via Hermitian inner product is fulfilled for the region of unbroken $\mathcal{PT}$ symmetry when it is taken the Hermitian limit, while for the region of broken $\mathcal{PT}$ symmetry the orthogonality is satisfied at the cost of coupling loss.

Assured the non-Hermiticity and the coupling of this model, the Hermitian inner product is meaningless as it can not provide the orthogonalization between the eigenstates of both regions of the Hamiltonian $\hat{H}$.

We are then led to the task of redefining the inner product. How can this be done? We start from what we already know. We know how the Hermitian inner product is done: By using the Hermitian symmetry. Given that in this model the Hermitian symmetry is superseded by the $\mathcal{PT}$ symmetry, it is fair to use it to redefine the inner product.

Let us remember how the Hermitian inner product is defined: Given two states $|u\rangle$ and $|v\rangle$, the Hermitian inner product between them is

$$\langle u | v \rangle \doteq |u\rangle^\dagger \cdot |v\rangle \,, \tag{2.1.12}$$

where $\dagger$, as we know, is the Hermitian operation.

In words, the Hermitian inner product between two states $|u\rangle$ and $|v\rangle$ is the matrix product between the states $|u\rangle^\dagger$ and $|v\rangle$, where $|u\rangle^\dagger$ is the Hermitian adjoint to the state $|u\rangle$.

Well, we created the $\mathcal{PT}$ inner product analogously to the Hermtian inner product: The $\mathcal{PT}$ inner product between two states $|u\rangle$ and $|v\rangle$ is the matrix product between the states $(\hat{U}_{\mathcal{PT}} |u\rangle)^\intercal$ and $|v\rangle$, where $(\hat{U}_{\mathcal{PT}} |u\rangle)^\intercal = \langle u|_{\mathcal{PT}}$ is the $\mathcal{PT}$ adjoint to the state[1] $|u\rangle$.

Mathematically, the $\mathcal{PT}$ inner product is defined as follows:

$$\langle u | v \rangle_{\mathcal{PT}} \doteq \left(\hat{U}_{\mathcal{PT}} |u\rangle \right)^\intercal \cdot |v\rangle = \langle u|_{\mathcal{PT}} |v\rangle \,. \tag{2.1.13}$$

[1] The transposition operation $\intercal$ was added to make the matrix product in the inner product defined below similar to that of Quantum Mechanics.

This is the ideal time to present an important result that allows us to stands for which parametric region the $\mathcal{PT}$ inner product is meaningful: The eigenstate of a Hamiltonian with symmetry $\mathcal{PT}$ has a real eigenvalue if it is also an eigenstate of the transformation $\mathcal{PT}$, represented by the operator $\hat{U}_{\mathcal{PT}}$ (BENDER, 2019, p. 40).

This result justifies the terminology used to describe regions of unbroken and broken $\mathcal{PT}$ symmetry. The first means that the eigenstates of the Hamiltonian are also eigenstates of the operator $\hat{U}_{\mathcal{PT}}$, which implies in the reality of the energy eigenvalues associated with these eigenstates. The second, on the other hand, reveals that the eigenstates of the Hamiltonian are not eigenstates of the operator $\hat{U}_{\mathcal{PT}}$, and consequently the energies eigenvalues associated with them are complex.

Therefore, the nomenclature used to distinguish the two regions of the energy spectrum into an unbroken and broken region concerns not only to the Hamiltonian, but rather whether the eigenstates of the Hamiltonian are simultaneously or not eigenstates of the operator $\hat{U}_{\mathcal{PT}}$. In effect, the Hamiltonian $\hat{H}$, e.g., exhibits symmetry $\mathcal{PT}$ in both regions regardless of the reality or complexity of the energy eigenvalues.

It is possible to show that the general form of the eigenstate of the operator $\hat{U}_{\mathcal{PT}}$ giving in (2.1.4) is

$$|\chi\rangle = \begin{pmatrix} w_2^* e^{i\theta} \\ w_2 \end{pmatrix} \quad \text{or} \quad |\chi\rangle = \begin{pmatrix} w_1 \\ w_1^* e^{i\theta} \end{pmatrix}. \tag{2.1.14}$$

One can see when comparing equations (2.1.6) and (2.1.14) that the states $|\psi_\pm\rangle$ and $|\chi\rangle$ have the same structure and they are the same as long as

$$\begin{cases} w_1 = w_1^* = w_2 = w_2^* = g; \\ \cos\theta = \dfrac{\sqrt{g^2 - b^2}}{g}, \quad \sin\theta = \dfrac{b}{g} \quad \Longrightarrow \quad g e^{i\theta} = \sqrt{g^2 - b^2} + ib. \end{cases} \tag{2.1.15}$$

Therefore, the simultaneous eigenstates of the operator $\hat{U}_{\mathcal{PT}}$ and the Hamiltonian $\hat{H}$ can be rewritten as follows:

$$|\psi_+\rangle = N \begin{pmatrix} g e^{i\theta} \\ g \end{pmatrix}, \quad |\psi_-\rangle = N \begin{pmatrix} -g \\ g e^{i\theta} \end{pmatrix}, \tag{2.1.16}$$

where this form alludes to a polar description of the eigenstates of this Hamiltonian.

In order to check the consistency of simultaneity of the states given in (2.1.16) as eigenstates of both operator $\hat{H}$ and $\hat{U}_{\mathcal{PT}}$, one can simply show that $\hat{H}\,|\psi_\pm\rangle = E_\pm\,|\psi_\pm\rangle$ and $\hat{U}_{\mathcal{PT}}\,|\psi_\pm\rangle = \pm e^{-i\theta}\,|\psi_\pm\rangle$, meaning that these new states are eigenstates of the both operators and have real energy eigenvalues, the same as the eigenvalues given in (2.1.6).

On the other hand, it can be seen that the eigenstates $|\phi_\pm\rangle$ of the region of broken $\mathcal{PT}$ symmetry are not compatible with the general form of the eigenstates of the operator $\hat{U}_{\mathcal{PT}}$ [compare equation (2.1.9) and equation (2.1.14)]. Actually, this comes from the observation that the components of $|\phi_\pm\rangle$ are not a set of a complex number and a pure real number as of $|\chi\rangle$, but instead one is a pure imaginary number and the other a real number.

In summary, it is not possible to rewrite $|\phi_\pm\rangle$ in polar form for an arbitrary angle θ; for $\theta = \pi/2, 3\pi/2$, the polar form is reproduced, but these values, according to (2.1.15), are equivalent to the points $g = \pm b$ in which the phase transitions of the $\mathcal{PT}$ symmetry occur, which are outside the range $g^2 < b^2$ where the symmetry is broken.

With this results, it is expected that the $\mathcal{PT}$ inner product is meaningful only for the region where the energy eigenvalues are real, that is, only for $|\psi_\pm\rangle$, since in this region the eigenstates of the Hamiltonian and the operator $\hat{U}_{\mathcal{PT}}$ are the same.[2]

Henceforward, we will only deal with the eigenstates belonging to the region of unbroken $\mathcal{PT}$ symmetry; the $\mathcal{PT}$ inner product is solely valid for these eigenstates and for the states spanning from them.

Now we can impose the normalization via $\mathcal{PT}$ inner product for the eigenstates of the region of unbroken $\mathcal{PT}$ symmetry, which is given in (2.1.16). It then can be rewritten as

$$|\psi_+\rangle = \frac{1}{\sqrt{2\cos\theta}}\begin{pmatrix} e^{i\theta} \\ 1 \end{pmatrix}, \quad |\psi_-\rangle = \frac{1}{\sqrt{2\cos\theta}}\begin{pmatrix} -1 \\ e^{i\theta} \end{pmatrix}, \tag{2.1.17}$$

so that

$$\langle\psi_\pm|\psi_\pm\rangle_{\mathcal{PT}} = \pm 1 \tag{2.1.18}$$

On the other hand, the orthogonality between these same eigenstates is satisfied when one uses the $\mathcal{PT}$ inner product:

$$\langle\psi_\pm|\psi_\mp\rangle_{\mathcal{PT}} = 0. \tag{2.1.19}$$

Based on the latest results, there appears that to be a way to define an inner product that reproduces the orthogonality between the eigenstates belonging to the region of unbroken $\mathcal{PT}$ symmetry of a non-Hermitian Hamiltonian.

However, the positivity of the norm is only partially satisfied, which is a problem. Certainly, to have a well-defined concept of probability in Quantum Mechanics, it is essential that the eigenstates of the Hamiltonian present orthonormality and positively defined norm relatively to an inner product.

---

[2] As a matter of fact, the norm of the eigenstates $|\phi_\pm\rangle$ vanishes when calculate via $\mathcal{PT}$ inner product.

Therefore, the following question arises: How to remove the negative sign from the norm $\langle\psi_-|\psi_-\rangle_{\mathcal{PT}}$?

Suppose it is possible to construct a commutable operator with $\mathcal{PT}$ and to which the state $|\psi_-\rangle$ is an eigenstate with an eigenvalue identical to the value of the norm $\langle\psi_-|\psi_-\rangle_{\mathcal{PT}}$. Therefore, if the $\mathcal{PT}$ inner product is again reformulated, but now by introducing this new operator, the negative value of the norm $\langle\psi_-|\psi_-\rangle_{\mathcal{PT}}$ becomes positive, making the normalization of the eigenstate $|\psi_-\rangle$ positively defined; evidently, the eigenstate $|\psi_+\rangle$ will also be an eigenstate of this new operator with eigenvalue given by the value of the norm $\langle\psi_+|\psi_+\rangle_{\mathcal{PT}}$, but the new inner product will not affect the normalization of this eigenstate because its norm is equal to unity.

It turns out, according to Bender (2019, p. 87-88), that this new operator whose action was previously described is a new symmetry of the Hamiltonian $\hat{H}$, existing only in the region of unbroken $\mathcal{PT}$ symmetry. This symmetry will be denoted by $\mathcal{N}$ due to its close relation with the value of the norm via the $\mathcal{PT}$ inner product.[3]

The operator $\hat{U}_{\mathcal{N}}$ of the symmetry $\mathcal{N}$ is defined as the sum of all tensor products between an eigenstate and its $\mathcal{PT}$ adjoint:

$$\hat{U}_{\mathcal{N}} \doteq |\psi_+\rangle \otimes \left(\hat{U}_{\mathcal{PT}}\,|\psi_+\rangle\right)^{\mathsf{T}} + |\psi_-\rangle \otimes \left(\hat{U}_{\mathcal{PT}}\,|\psi_-\rangle\right)^{\mathsf{T}} = |\psi_+\rangle\,\langle\psi_+|_{\mathcal{PT}} + |\psi_-\rangle\,\langle\psi_-|_{\mathcal{PT}}\,, \tag{2.1.20}$$

so that

$$\hat{U}_{\mathcal{N}}\,|\psi_\pm\rangle = \langle\psi_+|\psi_\pm\rangle_{\mathcal{PT}}\,|\psi_+\rangle + \langle\psi_-|\psi_\pm\rangle_{\mathcal{PT}}\,|\psi_\pm\rangle = \pm\,|\psi_\pm\rangle\,. \tag{2.1.21}$$

Observe, therefore, that the form of the operator $\mathcal{N}$ depends on the eigenstates of the Hamiltonian, meaning that quantum systems with different Hamiltonians will have different operators $\mathcal{N}$. Nevertheless, the way in which this operator is constructed resembles the closure relation of Hermitian models.

By using the equation (2.1.17) in (2.1.20) it can be shown that the explicit form of the operator $\mathcal{N}$ is given by

$$\hat{U}_{\mathcal{N}} = \frac{1}{\cos\theta}\begin{pmatrix} i\sin\theta & 1 \\ 1 & -i\sin\theta \end{pmatrix}, \tag{2.1.22}$$

and with this explicit form it possible to show that this operator has the following properties:

$$\hat{U}_{\mathcal{N}}^2 = 1 \qquad \Longrightarrow \qquad \hat{U}_{\mathcal{N}}^{-1} = \mathcal{N}. \tag{2.1.23}$$

[3] In other references, such as Bender (2019, sec. 3.6) or Alexandre, Ellis, and Millington (2020), this new transformation is denoted, respectively, as $\mathcal{C}$ or $\mathcal{C}'$, as it has a similar characteristic to the charge conjugation transformation; the "charge" that it counts is the sign of the normalization of the eigenstates via the $\mathcal{PT}$ inner product.

In words, this operator is an inversion whose eigenvalues are, at most, a complex phase.

Since the eigenstates of $\mathcal{N}$ and $\hat{H}$ are the same, both operators are expected to commute. Indeed, by using the equations (2.1.1), (2.1.15) and (2.1.22) it can be shown that

$$[\hat{U}_{\mathcal{N}}, \hat{H}] = 0. \tag{2.1.24}$$

that is, the operator $\mathcal{N}$ is a symmetry of the Hamiltonian $\hat{H}$.

Futhermore, it can be shown that the operator $\hat{U}_{\mathcal{N}}$ also commutes with the operator $\hat{U}_{\mathcal{PT}}$: $[\hat{U}_{\mathcal{N}}, \hat{U}_{\mathcal{PT}}] = 0$.

The last characteristic of the $\hat{U}_{\mathcal{N}}$ operator is that it reduces to the parity operator $\hat{U}_{\mathcal{P}}$ in the Hermitian limit. As a matter of fact, we note, according to (2.1.19), that the Hermitian limit $b = 0$ is equivalent to making $\sin\theta = 0$, i.e., $\theta = 0$. Then

$$\lim_{\theta\to 0}\hat{U}_{\mathcal{N}} = \lim_{\theta\to 0}\frac{1}{\cos\theta}\begin{pmatrix} i\sin\theta & 1 \\ 1 & -i\sin\theta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \hat{U}_{\mathcal{P}}. \tag{2.1.25}$$

We can now redefine the $\mathcal{PT}$ inner product with the help of $\mathcal{N}$ in order to have a positively defined norm: Analogously to the $\mathcal{PT}$ inner product, we just multiply, component by component, the $\mathcal{NPT}$ adjoint of a state, which is the action of $\mathcal{NPT}$ in that same state, by another state. Mathematically speaking, given two arbitrary states $|u\rangle$ and $|v\rangle$, we have the so-called $\mathcal{NPT}$ inner product:

$$\langle u|v\rangle_{\mathcal{NPT}} = \left(\hat{U}_{\mathcal{N}}\,|u\rangle\right)^{\mathsf{T}}\cdot|v\rangle = \langle u|_{\mathcal{NPT}}\,|v\rangle\,. \tag{2.1.26}$$

It is then possible, by using the $\mathcal{NPT}$ inner product, to show that the norm of the state $|\psi_\pm\rangle$ is positively defined and that the orthogonality between $|\psi_+\rangle$ e $|\psi_-\rangle$ is preserved:

$$\langle\psi_\pm|\psi_\pm\rangle_{\mathcal{NPT}} = 1, \quad \langle\psi_\pm|\psi_\mp\rangle_{\mathcal{NPT}} = 0. \tag{2.1.27}$$

Nevertheless, the following closure relation is valid

$$1 = |\psi_+\rangle\otimes\left(\hat{U}_{\mathcal{N}}\,|\psi_+\rangle\right)^{\mathsf{T}} + |\psi_-\rangle\otimes\left(\hat{U}_{\mathcal{N}}\,|\psi_-\rangle\right)^{\mathsf{T}} = |\psi_+\rangle\,\langle\psi_+|_{\mathcal{NPT}} + |\psi_-\rangle\,\langle\psi_-|_{\mathcal{NPT}}\,. \tag{2.1.28}$$

Thus, in the Hermitian limit $\theta\to 0$, the $\hat{U}_{\mathcal{NPT}}\to\hat{U}_{\mathcal{P}^2\mathcal{T}} = \hat{U}_{\mathcal{T}}$. This implies that the $\mathcal{CPT}$ symmetry reduces to the conventional condition of Hermiticity for a symmetric matrix (BENDER, 2019, p. 96).

The results in equations (2.1.27) and (2.1.28) allow us to state that the set formed by the eigenstates $|\psi_\pm\rangle$ of the Hamiltonian $\hat{H}$ is an orthonormal and complete basis under the

$\mathcal{NPT}$ inner product. With this basis and this inner product, we can reproduce the probabilistic interpretation intrinsically linked to Quantum Mechanics.

As the construction of the operator $\mathcal{N}$ depends on the eigenstates of the Hamiltonian $\hat{H}$, a direct consequence is that each non-Hermitian and $\mathcal{PT}$-symmetric system internal product, unlike Hermitian Quantum Mechanics whose internal product for all systems is the Hermitian one.

Furthermore, thanks to the dynamically constructed $\mathcal{NPT}$ inner product, the Hamiltonian, according to Beygi (2019, p. 6), becomes self-adjoint:

$$\langle u|\hat{H}v\rangle_{\mathcal{NPT}} = \langle \hat{H}u|v\rangle_{\mathcal{NPT}}, \tag{2.1.29}$$

where $|u\rangle$ e $|v\rangle$ arbitrary states. This arises from the property that $\hat{H}$ commutes with both operators $\hat{U}_{\mathcal{PT}}$ and $\mathcal{N}$, that is, it commutes with $\hat{U}_{\mathcal{NPT}}$.

Not only that, it can be demonstrated that an arbitrary state $|\Xi\rangle$, whose components are Ξ_1 and Ξ_2, has a positive norm via the inner product $\mathcal{NPT}$. In effect, knowing that the $\mathcal{NPT}$ adjoint to $|\Xi\rangle$ is

$$\hat{U}_{\mathcal{NPT}}\,|\Xi\rangle = \hat{U}_{\mathcal{NPT}}\begin{pmatrix}\Xi_1\\ \Xi_2\end{pmatrix} = \frac{1}{\cos\theta}\begin{pmatrix}\Xi_1^* + i\Xi_2\sin\theta\\ \Xi_2^* - i\Xi_1^*\sin\theta\end{pmatrix}, \tag{2.1.30}$$

the norm of this state via $\mathcal{NPT}$ inner product results

$$\langle\Xi|\Xi\rangle_{\mathcal{NPT}} = \frac{1}{\cos\theta}\big[|\Xi_1|^2 + |\Xi_2|^2 + i(\Xi_1\Xi_2^* - \Xi_1^*\Xi_2)\sin\theta\big], \tag{2.1.31}$$

or writing $\Xi_1 = w_1 + iz_1$ e $\Xi_2 = w_2 + iz_2$, with $w_1, w_2, z_1, z_2 \in \mathbb{R}$:

$$\langle\Xi|\Xi\rangle_{\mathcal{NPT}} = \frac{1}{\cos\theta}\big[w_1^2 + z_2^2 + 2w_1z_2\sin\theta + w_2^2 + z_1^2 - 2w_2z_1\sin\theta\big]. \tag{2.1.32}$$

This quantity, according to Bender (2019, p. 96), is positive and vanishes only for $w_1, w_2, z_1, z_2 = 0$.

It is worth mentioning that this two-level non-Hermitian $\mathcal{PT}$-symmetry model has applications in neutrino oscillations (OHLSSON, 2016; OHLSSON; ZHOU, 2020).

We finish this section citing another exactly solvable non-Hermitian and $\mathcal{PT}$-symmetric Hamiltonian: The Swanson Hamiltonian, whose properties were examined by Swanson (2004) and Graefe et al. (2015).

2.2 Non-Hermitian fermionic model with $\mathcal{PT}$ symmetry

The Lagrangian density of the usual free and massive fermionic field is given by[4]

$$\mathcal{L}_\psi = \bar{\psi}(x)(i\gamma^\alpha\partial_\alpha - m)\psi(x). \tag{2.2.1}$$

It is known that the previous Lagrangian density has symmetry $\mathcal{P}$ e $\mathcal{T}$. A simple way to see this is to remember that the equation of motion for the field ψ, derived from the Euler-Lagrange equations, is covariant. Thus, the covariance of the equation of motion guarantees that any transformation belonging to the Lorentz group, which is the case of $\mathcal{P}$ and $\mathcal{T}$, will be symmetry of this model.

On the other hand, this Lagrangian density is not Hermitian:

$$\mathcal{L}_\psi^\dagger = \mathcal{L}_\psi - i\partial_\alpha\big[\bar{\psi}(x)\gamma^\alpha\psi(x)\big], \tag{2.2.2}$$

in which the remaining term in brackets is called boundary term, which does not contributes to the field equations.

However, the action generated by $\mathcal{L}_\psi$, which is given by

$$S_\psi = \int \mathrm{d}^4x\, \mathcal{L}_\psi, \tag{2.2.3}$$

will be evaluated at its boundary which the infinity. Therefore, assuming that the field is regular and vanish at the infinity, the boundary term in the Lagrangian density is negligible in action. This ensures that the action is real and the model is said to be Hermitian.

Thus it is possible to say that Hermiticity is expressed in the Lagrangian density if it remains invariant under Hermitian conjugation at less than a boundary term.

Our sought extension must violate Hermiticity while preserving $\mathcal{PT}$ symmetry without adding interactions. With this in mind, it can be recalled that the axial current $j_5 = \bar{\psi}\gamma_5\psi$ receives a negative sign when act under both the parity and time inversion transformations. Therefore, this current is globally invariant under $\mathcal{PT}$ (DAS, 2021, p. 432 and 471).

Thus a possible extension to a $\mathcal{L}_\psi$ that exhibits $\mathcal{PT}$ symmetry is $\mathcal{L}_\psi \to \mathcal{L}_\psi - \mu j_5$:

$$\mathcal{L}_{\mathrm{E}\psi} = \bar{\psi}(x)(i\gamma^\alpha\partial_\alpha - m - \mu\gamma_5)\psi(x), \quad \mu \in \mathbb{R}, \tag{2.2.4}$$

where the subscript “E” refers to the extension.

[4] The natural units are used throughout this dissertation.

The Hermitian operation on the Lagrangian density $\mathcal{L}_{\mathrm{E}\psi}$, given that $j_5^\dagger = -j_5$, gives the following result:

$$\mathcal{L}_{\mathrm{E}\psi}^\dagger = \mathcal{L}_{\mathrm{E}\psi} - i\partial_\alpha\left[\bar{\psi}(x)\gamma^\alpha\psi(x)\right] + 2\mu\bar{\psi}(x)\gamma_5\psi(x). \tag{2.2.5}$$

The second term of equation (2.2.5) is equal to the boundary term obtained in equation (2.2.2), and is therefore negligible when evaluating the Hermiticity of the model. The last term, however, is not a boundary term and, therefore, does not vanishes in the integration of the action. So it is possible to conclude that the Hermiticity of the model is transgressed.

From this result, the parameter μ can be seen as the parameter that controls the violation of the Hermiticity of the model: For $\mu \neq 0$, we have a non-Hermitian model; For $\mu = 0$, the Hermiticity is restored.

Now one may ask the following: To what values is the parameter μ restricted for the energy spectrum of this model to be real? To answer this question it is first necessary to obtain the energy spectrum.

In order to achieve this, firstly we need the equation of motion, which is obtained from (2.2.4):

$$(i\gamma^\alpha\partial_\alpha - m - \mu\gamma_5)\psi(x) = 0. \tag{2.2.6}$$

The equation (2.2.6), however, can be rewritten in a convenient way by applying on the left the operator $\left(-i\gamma^\beta\partial_\beta - m + \mu\gamma_5\right)$:

$$\left(\Box + m^2 - \mu^2\right)\psi(x) = 0, \tag{2.2.7}$$

where $\left(\gamma^5\right)^2 = 1$, $\{\gamma_5, \gamma^\alpha\} = 0$ and $\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}$ were used.

We can now rewrite $\psi(x)$ in moment space:

$$\psi(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\tilde{\psi}(p)e^{-ip\cdot x} \tag{2.2.8}$$

and replace in equation (2.2.7), obtaining as result

$$(-p^2 + m^2 - \mu^2)\tilde{\psi}(p) = 0, \tag{2.2.9}$$

whose solution for a general $\tilde{\psi}(p)$ field must satisfies

$$p^2 = M^2 \doteq m^2 - \mu^2, \tag{2.2.10}$$

which is the dispersion relation of the model; note that the square of the effective mass of the model is $M^2 = m^2 - \mu^2$, i.e., μ can be interpreted as a mass parameter.[5]

[5] As μ is accompanied by the axial current $j_5 = \bar{\psi}\gamma_5\psi$, which behaves like a pseudoscalar, a possible cognomen for this term is pseudomass or axial mass.

An expression for the energy can finally be obtained by remembering that $p^2 = E^2 - |\mathbf{p}|^2$ and then use it in equation (2.2.10), which yields

$$E = \pm\sqrt{|\mathbf{p}|^2 + M^2}. \tag{2.2.11}$$

It can be seen in the equation (2.2.11) that the energy is real, for any value of momentum $\mathbf{p}$, if $M^2 \geq 0$, i.e., $m^2 \geq \mu^2$. On the other hand, the energy is not necessarily real and can be complex depending on the value of the momentum $\mathbf{p}$ if $m^2 \leq \mu^2$. Finally, it may be mentioned that the limit for a non-massive model is reached when the effective mass is null, i.e., $M^2 = 0$, which is equivalent to consider $\mu = \pm m$. Thus the non-massive model can be described without the scalar m and axial μ masses being necessarily zero.

We assert that in this model the axial mass μ controls no only the Hermiticity and the reality of the energy spectrum, but also the predominance of which chirality the fermionic field predominantly assumes.

This can be seen from the conserved current density of this model, which can be obtained from the equation of motion (ALEXANDRE; BENDER, 2015), and is given by

$$j^{\alpha}_{\mathrm{E}}(x) = \bar{\psi}(x)\gamma^{\alpha}\Big(1 + \frac{\mu}{m}\gamma_5\Big)\psi(x). \tag{2.2.12}$$

It has an axial deformation proportional to the ratio between the masses of the model. Therefore, the interaction of a fermion with a gauge field, such as the electromagnetic field, is expected to be deformed.

On the other hand, it is well-known that the γ_5 matrix is present in the chirality projection operators of the fermionic field; this justifies our previous statement. Thus conserved current density alludes to the fact that this extended model can distinguish the chiralities of this field.

To see explicitly how the axial mass μ selects the chirality of the fermionic field, it is necessary to rewrite the conserved current density in such a way that this field is expressed in terms of its left-hand and right-hand chiralities. This can be realized by expressing the fermionic field in the chiral basis

$$\psi(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix}, \tag{2.2.13}$$

where the components are given in terms of the projection operators

$$P_L = \frac{1}{2}(1 - \gamma_5) \quad \text{e} \quad P_R = \frac{1}{2}(1 + \gamma_5) \tag{2.2.14}$$

through the following relation:

$$\psi_L(x) = P_L\psi(x) \quad \text{e} \quad \psi_R(x) = P_R\psi(x). \tag{2.2.15}$$

Here, we consider the Dirac matrices and the γ_5 matrix in the Weyl representation:

$$\gamma^\alpha = \begin{pmatrix} 0 & \sigma^\alpha \\ \bar{\sigma}^\alpha & 0 \end{pmatrix} \quad \text{e} \quad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.2.16}$$

where

$$\sigma^\alpha = (1, \boldsymbol{\sigma}) \quad \text{e} \quad \bar{\sigma}^\alpha = (1, -\boldsymbol{\sigma}), \tag{2.2.17}$$

with $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ being the Pauli vector, which is composed by the matrices.

Therefore, with the help of equation (2.2.13), the conserved current density given in equation (2.2.12) can be rewritten as (ALEXANDRE; BENDER; MILLINGTON, 2017)

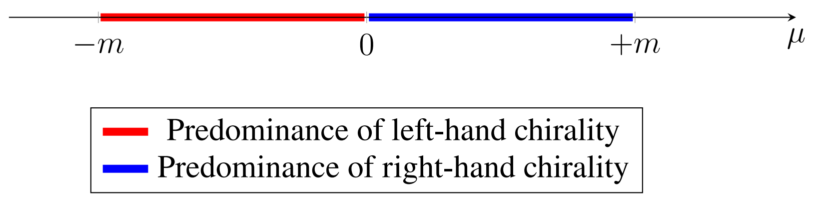
$$j^\alpha(x) = \psi_L^\dagger(x)\bar{\sigma}^\alpha\psi_L(x)\Big(1 - \frac{\mu}{m}\Big) + \psi_R^\dagger(x)\sigma^\alpha\psi_R(x)\Big(1 + \frac{\mu}{m}\Big). \tag{2.2.18}$$

It can be seen, then, that the limits of a massless model $\mu = \pm m$ are equivalent to limiting the field to a single chirality: For $\mu = m$, we have a model whose conserved current density is composed exclusively of the right-hand chirality of the field, which means that the field interacts only through this chirality; For $\mu = -m$, on the other hand, it is composed of the left-hand chirality of the field, which again the means that the field interacts only through a single chirality.

Furthermore, in model with gauge symmetry the conserved current density is responsible for the interaction between the fermion and the gauge field. We then can concluded that, at least in the non-massive limits, the gauge field interacts only with a single chirality of the field, which is essential for the description of neutrinos and antineutrinos, as these are described by fields with only one type of chirality (left- and right-hand, respectively) (BILENKY, 2018).

The figure 2.2 illustrates the fermionic field of this model behaves in different ways depending on the values assumed by the parameter μ.

Figure 2.2 – Behavior of the fermionic field $\psi(x)$ in terms of the axial mass parameter μ. At $\mu = -m$, we have a (massless) Weyl fermion with exclusively left-handed chirality; in the interval $-m < \mu < 0$, in red, there is a fermion whose predominant chirality is left-handed; at $m = 0$, the Hermitian limit is obtained, recovering the Dirac fermion; in the interval $0 < \mu < m$, in blue, there is a fermion whose predominant chirality is right-handed; at $\mu = m$, finally, we have a Weyl fermion (not massive) but with strictly right-handed chirality.



Source: Authors (2024), based on Alexandre, Bender, and Millington (2017, p. 7, Figure 1).

3 NON-RELATIVISTIC REGIME OF NON-HERMITIAN QED

This chapter will deal with the main aspect of this dissertation: The non-relativistic regime of the non-Hermitian QED. This regime is taken for a scattering process in which a non-Hermitian fermion is scattered off by a classical external electromagnetic field. The final result is the well-known Pauli-Schrödinger Hamiltonian.

We therefore expect that non-Hermitian corrections can contribute to phenomenological aspects of the non-relativistic regime, where new couplings may emerge.

For didactic purposes, the Pauli-Schrödinger Hamiltonian for the usual QED is revisited in section 3.1. It is obtained from two different approach: (i) The non-relativistic limit taken for the Dirac equation, which is presented in section 3.1.1; and (ii) The non-relativistic limit taken for the tree-level QED, which is presented in section 3.1.2. We will observe that these two procedures give the same Hamiltonian for the usual fermionic model.

In section 3.2 we follow the same two aforementioned procedures, but now for a fermion with axial mass, and in section 3.3 for a fermion with axial mass and V-A coupling to the electromagnetic field. Unlike the usual fermionic model, the two procedures do not give the same Hamiltonian; the non-relativistic limit taken for the tree-level QED seems to be more complete.

In the section 3.4 we discuss some phenomenological aspects about the new couplings obtained for the non-Hermitian fermionic models and establishe approximate values of a supposed axial mass for the electronic neutrino.

The section 3.5 intends to present a application for our results: Find out the hydrogen atom spectrum for the Hamiltonian obtained in section 3.3.

3.1 Pauli-Schrödinger Hamiltonian in the usual QED

Before we begin our analysis for the non-Hermitian QED, let us review the two distinct ways of arriving at the Pauli-Schrödinger Hamiltonian, one using the usual Dirac theory and one using the usual QED. This will be useful at the time of making comparison between our new results and these well-established from the usual theory.

3.1.1 First approach: canonical Dirac equation

In this subsection we aimed to obtain the Pauli-Schrödinger Hamiltonian by taking directly the non-relativistic limit of the Dirac equation.

The canonical form of the Dirac equation under minimal coupling is given by[1]

$$E\psi = (\boldsymbol{\alpha} \cdot [\mathbf{p} - e\mathbf{A}] + e\phi + \beta m)\psi, \tag{3.1.1}$$

which describes a Dirac fermion field $\psi(x)$ with mass m, charge e, and energy E coupled to an electromagnetic field $(\phi, \mathbf{A})$.

Now we wish to consider the non-relativistic limiting case, for which one obtains

$$\begin{cases} E\varphi = (e\phi + m)\varphi + [\boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A})]\chi, \\ E\chi = [\boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A})]\varphi + (e\phi - m)\chi, \end{cases} \tag{3.1.2}$$

in which were used the two-spinor decomposition

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \tag{3.1.3}$$

and also the Dirac representation for the $\boldsymbol{\alpha}$ and β matrices:

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \tag{3.1.4}$$

where $\boldsymbol{\sigma}$ is composed by the Pauli matrices.

Moreover, it is possible rewrite (3.1.2) so that one gets for the upper component φ the eigenvalue equation $G_m\varphi = H_{\text{P-S}}\varphi$, in which $H_{\text{P-S}}$ is the Pauli-Schrödinger Hamiltonian and it is given by

$$H_{\text{P-S}} = \left(e\phi + \frac{[\boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A})]^2}{E + m - e\phi} \right). \tag{3.1.5}$$

It is worth to note that $G_m = E - m$ is identified as the non-relativistic energy, since it is the energy at which the rest energy m has been removed from the total energy E.

The non-relativistic regime is achieved by considering $H_{\text{P-S}}$ under the condition of $\mathbf{v}^2 \ll 1$, for which one gets the following approximations: (i) $E = \sqrt{\mathbf{p}^2 + m^2} \approx m$ and also (ii) That the electric potential energy $e\phi$ has at most a magnitude in the range of the kinetic energy, i.e., $e\phi \approx \frac{1}{2}m\mathbf{v}^2 \ll m$.

[1] The natural units are used throughout this dissertation.

Hence, under these approximations $H_{\text{P-S}}$ is cast in the following form:

$$H_{\text{P-S}} = \left(e\phi + \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 - \frac{e}{2m}(\boldsymbol{\sigma} \cdot \mathbf{B})\right), \tag{3.1.6}$$

where was used the identity $[\boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A})]^2 = (\mathbf{p} - e\mathbf{A})^2 - e(\boldsymbol{\sigma} \cdot \mathbf{B})$.

In order to investigate the couplings to electric potential and to the magnetic field we restrict our analyzes to constant magnetic fields $\mathbf{A} = \frac{1}{2}(\mathbf{B} \times \mathbf{r})$, which in the weak field approximation implies

$$H_{\text{P-S}} = \frac{\mathbf{p}^2}{2m} + e\phi - \frac{e}{2m}(\mathbf{L} + 2\mathbf{S}) \cdot \mathbf{B}, \tag{3.1.7}$$

where we have recognized the spin operator $\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}$ and the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$.

One can now establish physical insights about the couplings contained in the Pauli-Schrödinger Hamiltonian (3.1.7).

In the first point, we note that the electric charge couples via the electric potential $e\phi$. We then define the electrical coupling as the quantity that keeps up with the electric potential and we denote it as g_E. As can be seen from (3.1.7), for the usual Dirac equation we have just[2]

$$g_E^{\text{Dirac}} = e. \tag{3.1.8}$$

As a second point, we have the couplings of angular momentum and spin to the magnetic field. However, it is well-established that the interaction energy between constant magnetic fields and angular momentum and spin is given by

$$U_{LS} = -\frac{e}{2m}(g_L\mathbf{L} + g_S\mathbf{S}) \cdot \mathbf{B}, \tag{3.1.9}$$

where g_L and g_S are the g-factors of the angular moment and spin, respectively. Then these g-factors are the quantities coupling to $\mathbf{L} \cdot \mathbf{B}$ or $\mathbf{S} \cdot \mathbf{B}$ besides the half charge-mass ratio $e/2m$, which we can refer as the magnetic coupling[3]:

$$g_B^{\text{Dirac}} = \frac{e}{2m}. \tag{3.1.10}$$

As it can be seen from (3.1.7), for the usual Dirac theory these g-factors are given by[4]

$$\begin{aligned} g_L^{\text{Dirac}} &= 1, \\ g_S^{\text{Dirac}} &= 2. \end{aligned} \tag{3.1.11}$$

[2] The sign of the electric coupling is defined as being the same as that of the electrical potential term in the Hamiltonian.

[3] The sign of the magnetic coupling is defined as being the opposite as that of the interaction term between magnetic field and angular momentum or spin in the Hamiltonian.

[4] The signs of the angular momentum and spin g-factors are defined as being the same as that of the magnetic coupling.

The prediction of value 2 for the spin g-factor was one the most important achievement made by the Dirac theory [5] (JEGERLEHNER, 2017, p. 7).

The couplings obtained by taking the non-relativistic limit directly from the Dirac equation are summarized in table 3.1.

Table 3.1 – Couplings from the usual Dirac theory.

Couplings	Values
Electric	$g_E^{\text{Dirac}} = e$
Magnetic	$g_B^{\text{Dirac}} = \frac{e}{2m}$
Angular momentum	$g_L^{\text{Dirac}} = 1$
Spin	$g_S^{\text{Dirac}} = 2$

Source: Authors (2024).

3.1.2 Second approach: tree-level QED

In the section 3.1.1, we arrive out to the Pauli-Schrödinger Hamiltonian just by considering the non-relativistic limit of the Dirac equation. Now we take a step forward and consider the QFT framework, particularly the QED.

In this case we can use the Feynman rules to calculate the tree-level scattering of a fermion by a classical external electromagnetic field. The final result, after taking again the non-relativistic limit, known as the Born approximation, is exactly the same Pauli-Schrödinger Hamiltonian obtained from the non-relativistic limit of the Dirac equation.

However, although both approaches lead to the same Pauli-Schrödinger Hamiltonian, the situation is different when the non-Hermitian QED is considered. Actually, as will be seen in section 3.2 and section 3.3, the Born approximation gives a more general structure for the Hamiltonian: It gives contributions due the presence of axial mass, which does not occur for the Dirac equation. We return to this point later.

[5] It is worth mentioning that the spin g-factor of fermions deviates a little from the value 2. This deviation is known as anomalous magnetic moment. For the electron, there is a match between experimental measurement and the Standard Model, c.f. Fan et al. (2023). For the muon, on the other hand, there is no such match, c.f. Aguillard et al. (2023).

The QED Lagrangian density under the presence of an external electromangetic field is given by

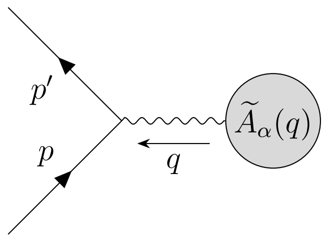
$$\mathcal{L} = \bar{\psi}(i\gamma^{\alpha}\partial_{\alpha} - m)\psi - e\bar{\psi}\gamma^{\alpha}\psi A_{\alpha}, \tag{3.1.12}$$

where ψ is the fermionic field and A_{α} is the external electromagnetic field. From this Lagrangian density one obtains the interaction Hamiltonian

$$H_{\text{int}} = \int \mathrm{d}^3x\, e\bar{\psi}(x)\gamma^{\alpha}\psi(x)A_{\alpha}(x). \tag{3.1.13}$$

The Born approximation consists in treating the gauge field as a classical potential $A_{\alpha}(x)$ in such a way that the fermion will be scattered by a photon emitted by the external gauge field. The corresponding Feynman diagram can be seen in figure 3.1.

Figure 3.1 – Fermion with initial momentum p and a given spin polarization is scattered off by a classical electromagnetic potential $\widetilde{A}_{\alpha}(q)$, where $q = p' - p$ is the transferred moment, to a state with final momentum p' and same initial spin polarization.



Source: Authors (2024).

The scattering total process is given by the S-matrix:

$$\langle p_{\text{out}}|p_{\text{in}}\rangle = \langle p'|S|p\rangle = \langle p'|p\rangle + \langle p'|iT|p\rangle\,, \tag{3.1.14}$$

where T is the transfer matrix, which concerns only the interaction between the fermion and the classical electromagnetic field, i.e., describes deviations from the free theory.

In terms of the interaction Hamiltonian one finds, at the leading order, that

$$\langle p'|iT|p\rangle = -ie\bar{u}(p')\gamma^{\mu}u(p)\widetilde{A}_{\mu}(p'-p). \tag{3.1.15}$$

Let us now consider a time-independent electromagnetic field $A(x) = A(\mathbf{x})$ in such a way that the scattering matrix can be written as

$$i\mathcal{M} = -ie\bar{u}(p')\gamma^{\mu}u(p)\widetilde{A}_{\mu}(\mathbf{q}), \tag{3.1.16}$$

where we have used the definition of the scattering matrix $\langle p'|iT|p\rangle = i(2\pi)\delta\left(p'_0 - p_0\right)\mathcal{M}$ (PESKIN; SCHROEDER, 1995, cf. exercise 4.4 (b)), which has a δ-function enforcing the energy conservation.

We then arrived in an expression for the scattering matrix $\mathcal{M}$ at the leading order contribution for a fermion scattered by a photon emitted by a classical electromagnetic field.

In order to evaluate the scattering matrix equation (3.1.16) it is necessary to determine the spinor solution $\psi(x) = u(p)e^{-ip\cdot x}$ for the Dirac equation $(i\gamma^\alpha\partial_\alpha - m)\psi(x) = 0$. In the Weyl representation of γ-matrices the solution reads (PESKIN; SCHROEDER, 1995, p. 46)

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix}, \tag{3.1.17}$$

where $\sigma^\alpha = (1, \boldsymbol{\sigma})$ and $\bar{\sigma}^\alpha = (1, -\boldsymbol{\sigma})$, and ξ is a unitary 2-vector which satisfies $\xi^\dagger\xi = 1$.

Moreover, in order to consider the non-relativistic limit of the scattering matrix equation (3.1.16), we should expand the spinor solution (3.1.17) up to the leading order [$\mathcal{O}(m^{-1})$]

$$u(p) = \sqrt{m}\begin{pmatrix} (1 - \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{2m})\xi \\ (1 + \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{2m})\xi \end{pmatrix}, \tag{3.1.18}$$

where the approximation $E \approx m$ was used.

In terms of these considerations, we can evaluate the relevant pieces of (3.1.16):

$$\bar{u}(p')\gamma^0 u(p) = 2m\xi'^\dagger\xi, \tag{3.1.19a}$$

$$\bar{u}(p')\gamma^k u(p) = 2m\xi'^\dagger\left(\frac{p'^k + p^k}{2m} - \frac{i}{2m}\epsilon^{ijk}q^i\sigma^j\right)\xi. \tag{3.1.19b}$$

Hence, carrying out the results (3.1.19a) and (3.1.19b) into (3.1.16), one founds the following relation:

$$i\mathcal{M}_{nr} = -i\Big\{(\xi'^\dagger\xi)e\widetilde{\phi}(\mathbf{q}) - \frac{e}{2m}\xi'^\dagger\Big[(\mathbf{p}' + \mathbf{p})\cdot\widetilde{\mathbf{A}}(\mathbf{q})\Big]\xi - \frac{e}{2m}(\xi'^\dagger\boldsymbol{\sigma}\xi)\cdot\widetilde{\mathbf{B}}(\mathbf{q})\Big\}, \tag{3.1.20}$$

in which three steps were made: (i) The Fourier transform of the four-potential was written down as $\widetilde{A}_\alpha(\mathbf{q}) = (\widetilde{\phi}(\mathbf{q}), -\widetilde{\mathbf{A}}(\mathbf{q}))$, (ii) The Fourier transform of the magnetic field was recognized as $\widetilde{B}^j(\mathbf{q}) = -i\epsilon^{ijk}q^i\widetilde{A}^k(\mathbf{q})$ and (iii) It was defined the scattering matrix non-relativistically normalized: $\mathcal{M}_{nr} = \mathcal{M}/2m$.

Now remembering that the non-relativistic scattering amplitude is related with a scattering potential $V(\mathbf{r})$ in terms of the Born approximation,

$$i\mathcal{M}_{nr} = -i\,\langle p'|V(\mathbf{r})|p\rangle = -i\int \mathrm{d}^3x\, V(\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}, \tag{3.1.21}$$

the scattering amplitude (3.1.20) corresponds to a scattering potential $V(\mathbf{r})$ given by

$$V(\mathbf{r}) = e(\xi'^{\dagger}\xi)\phi(\mathbf{r}) - \frac{e}{2m}\xi'^{\dagger}[(\mathbf{p}' + \mathbf{p}) \cdot \mathbf{A}(\mathbf{r})]\xi - \frac{e}{2m}(\xi'^{\dagger}\boldsymbol{\sigma}\xi) \cdot \mathbf{B}(\mathbf{r}). \tag{3.1.22}$$

At last, the Pauli-Schrödinger Hamiltonian can be determined by considering: (i) The low-energy regime,[6] (ii) The condition $\xi'^{\dagger}\xi = 1$, which stands for same values of initial and final spin polarizations, (iii) The magnetic field constant for which stands $\mathbf{A} = \frac{1}{2}(\mathbf{B} \times \mathbf{r})$ and (iv) Add up the kinetic energy $\frac{\mathbf{p}^2}{2m}$. Taking all these aspects into consideration, one obtains

$$H_{\text{P-S}} = \frac{\mathbf{p}^2}{2m} + e\phi - \frac{e}{2m}(\langle\mathbf{L}\rangle + 2\langle\mathbf{S}\rangle) \cdot \mathbf{B}, \tag{3.1.23}$$

where

$$\langle\mathbf{L}\rangle = \xi'^{\dagger}\mathbf{L}\xi \quad \text{and} \quad \langle\mathbf{S}\rangle = \xi'^{\dagger}\mathbf{S}\xi \tag{3.1.24}$$

are the expected values of the angular momentum and spin, respectively.

A quick look into equations (3.1.7) and (3.1.23) allow us to conclude that they are the same, that is, the description of the interaction between a fermion and the classical electromagnetic field by the Dirac equation corresponds to the lowest order contribution of QED.

The couplings obtained by taking the non-relativistic limit from the tree-level QED are summarized in table 3.2.

Table 3.2 – Couplings from the QED.

Couplings	Values
Electric	$g_E^{\text{QED}} = e$
Magnetic	$g_B^{\text{QED}} = \frac{e}{2m}$
Angular momentum	$g_L^{\text{QED}} = 1$
Spin	$g_S^{\text{QED}} = 2$

Source: Authors (2024).

3.2 Pauli-Schrödinger Hamiltonian in QED with fermionic axial mass

Now we will consider a non-Hermitian extension of the Dirac equation by only introducing an axial mass term as $\mu\gamma_5$, where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and μ is a real parameter referred as axial

[6] A regime in which the transferred momentum of photon becomes very small, i.e., $\mathbf{q} \to 0$, which implies $\mathbf{p}' = \mathbf{p}$.

mass; the minimal coupling stands unchanged. One can, therefore, follow the two procedures presented in sections 3.1.1 and 3.1.2, and compare these new results with the results from the usual theory.

3.2.1 First approach: canonical Dirac equation with fermionic axial mass

In this subsection we will obtain the Pauli-Schrödinger Hamiltonian from the canonical Dirac equation with a small deviation: we change the ordinary mass m of the free-fermion field $\psi(x)$ to an effective mass

$$M = \sqrt{m^2 - \mu^2}, \tag{3.2.1}$$

where $\mu \in \mathbb{R}$ and $-m \leq \mu \leq m$ is the restriction to the axial mass in order to the effective mass M be real-valued [for details, see the discussion on the section 2.2].

This is achieved by modifying the canonical equation with a term $\mu\gamma_5$ (BENDER; JONES; RIVERS, 2005; ALEXANDRE; BENDER, 2015), which has the following form when written as an energy eigenvalue equation:

$$E\psi(x) = (\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m + \beta\gamma_5\mu)\psi(x). \tag{3.2.2}$$

By following the same procedure as that one present in subsection 3.1.1 with the same minimal coupling, by using the Dirac representation for $\boldsymbol{\alpha}$, β [cf. equation (3.1.4)] and γ_5, which is given by

$$\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.2.3}$$

and by splitting the fermion field $\psi(x)$ in the two-spinor representation given in equation (3.1.3), we get

$$\begin{cases} E\varphi = (e\phi + m)\varphi + [\boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A}) + \mu]\chi, \\ E\chi = [\boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A}) - \mu]\varphi + (e\phi - m)\chi. \end{cases} \tag{3.2.4}$$

Moreover, we can solve the second equation in (3.2.4) for χ and then use it in the first one to get

$$E\varphi = \left\{ e\phi + m + \frac{[\boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A})]^2 - \mu^2}{E + m - e\phi} \right\}\varphi. \tag{3.2.5}$$

In the non-relativistic limit we can assume that the electric potential $e\phi$ has at most a magnitude in the range of kinetic energy, i.e, $e\phi \approx M\mathbf{v}^2/2$. However, in this limit $\mathbf{v}^2 \ll 1$, then $e\phi \ll M$. On the other hand, the total energy in this same limit is approximately the effective mass, $E \approx M$, which in turn can be approximate to $M \approx m - \mu^2/2m$, for $\mu \ll m$. Then, the denominator in the rightmost term in the right-hand side of the equation (3.2.5) can be estimated as

$$E + m - e\phi \approx 2m\big[1 + \mathcal{O}(m^{-2})\big]. \tag{3.2.6}$$

By neglecting terms as $\mathcal{O}(m^{-2})$, we get

$$G_M\varphi = H_{\text{P-S}}\varphi, \qquad G_M = E - \left(m - \frac{\mu^2}{2m}\right) \approx E - M, \tag{3.2.7}$$

where G_M is the non-relativistic energy and $H_{\text{P-S}}$, after using the identity $[\boldsymbol{\sigma}\cdot(\mathbf{p} - e\mathbf{A})]^2 = (\mathbf{p}-e\mathbf{A})^2 - e(\boldsymbol{\sigma}\cdot\mathbf{B})$, is the same as in (3.1.6), that is, is the usual Pauli-Schrödinger Hamiltonian.

In case of constant magnetic fields, the same result given in equation (3.1.7) is reached, namely

$$H_{\text{P-S}} = \frac{\mathbf{p}^2}{2m} + e\phi - \frac{e}{2m}(\mathbf{L} + 2\mathbf{S})\cdot\mathbf{B}, \tag{3.2.8}$$

which allows us to conclude naively the following: the electric, the spin, and the angular momentum couplings between a fermion with axial mass and the electromagnetic field do not receive any corrections compared to a usual fermion.

The couplings obtained by taking the non-relativistic limit directly from the Dirac equation with the fermionic axial mass are summarized in table 3.3.

Table 3.3 – Couplings from the Dirac theory with fermionic axial mass.

Couplings	Values
Electric	$g_E^{\text{AM-Dirac}} = e$
Magnetic	$g_B^{\text{AM-Dirac}} = \frac{e}{2m}$
Angular momentum	$g_L^{\text{AM-Dirac}} = 1$
Spin	$g_S^{\text{AM-Dirac}} = 2$

Source: Authors (2024).

3.2.2 Second approach: tree-level QED with fermionic axial mass

In this subsection it will be seen that for the model with fermionic axial massa there is a disagreement between the results of the two approaches, namely the non-relativistic limit of the Dirac equation and the non-relativistic limit of tree-level QED. Furthermore, it will be shown that there are corrections in the couplings when this model is compared with the usual one.

The Lagrangian density for the QED with fermionic axial mass is

$$\mathcal{L} = \bar{\psi}(i\gamma^\alpha\partial_\alpha - m - \mu\gamma_5)\psi - e\bar{\psi}\gamma^\alpha\psi A_\alpha. \tag{3.2.9}$$

Looking at the interaction term in equation (3.2.9), we can quickly conclude that it is the same as the usual QED, so the scattering amplitude $\mathcal{M}$ for this fermionic model with axial mass will be equal to of the usual model. The difference lies in the spinorial solution of the fermionic field $\psi(x)$, which will now be the solution of the Dirac equation with axial mass.

Now let us restate the scattering amplitude of a fermion scattered by a classical electromagnetic field, which is given in equation (3.1.16):

$$i\mathcal{M} = -ie\bar{u}(p')\gamma^\mu u(p)\widetilde{A}_\mu(\mathbf{q}), \tag{3.2.10}$$

As before we need the spinorial solution $\psi(x) = u(p)e^{-ip\cdot x}$, but now for the Dirac equation with axial mass, which is

$$(i\gamma^\alpha\partial_\alpha - m - \mu\gamma_5)\psi(x) = 0. \tag{3.2.11}$$

The solution in the Weyl representation of γ-matrices turns out to be

$$u(p) = \frac{1}{\sqrt{M}}\begin{pmatrix}\sqrt{m-\mu}\sqrt{p\cdot\sigma}\xi \\ \sqrt{m+\mu}\sqrt{p\cdot\bar{\sigma}}\xi\end{pmatrix}, \tag{3.2.12}$$

where ξ is a unitary 2-vector which satisfies $\xi^\dagger\xi = 1$, in which was used the representation

$$\gamma^5 = \begin{pmatrix}-1 & 0 \\ 0 & 1\end{pmatrix}. \tag{3.2.13}$$

However, we obtain the Pauli-Schrödinger Hamiltonian by taking the non-relativistic limit, and this limit for the spinor solution (3.2.12) is achieved by expanding the momentum square root to $\mathcal{O}(M^{-2})$ and setting $E \approx M$:

$$u(p) = \begin{pmatrix}\sqrt{m+\mu}\big(1 - \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{2M}\big)\xi \\ \sqrt{m-\mu}\big(1 + \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{2M}\big)\xi\end{pmatrix}. \tag{3.2.14}$$

Now some straightforward calculations give, by neglecting terms of $\mathcal{O}(M^{-2})$,

$$\bar{u}(p')\gamma^0 u(p) = 2M\xi'^\dagger\Big[\frac{m}{M} - \frac{\mu}{2M^2}(\mathbf{p}+\mathbf{p}')\cdot\boldsymbol{\sigma}\Big]\xi, \tag{3.2.15a}$$

$$\bar{u}(p')\gamma^k u(p) = 2M\xi'^\dagger\left\{\frac{m\big[(p'+p)^k - i\epsilon^{ijk}q^i\sigma^j\big]}{2M^2} - \frac{\mu\sigma^k}{M}\right\}\xi. \tag{3.2.15b}$$

Moreover, we use equations (3.2.15a) and (3.2.15b) to calculate the scattering matrix $\mathcal{M}$ given in (3.2.10):

$$\begin{aligned} i\mathcal{M}_{nr} = -i \Big\{ \Big[\frac{m}{M}(\xi'^{\dagger}\xi)\Big] e\widetilde{\phi}(\mathbf{q}) - \Big[\frac{\mu}{M^2}\xi'^{\dagger}[(\mathbf{p}'+\mathbf{p})\cdot\boldsymbol{\sigma}]\xi\Big] e\widetilde{\phi}(\mathbf{q}) \\ -\frac{em}{2M^2}\Big[\xi'^{\dagger}[(\mathbf{p}'+\mathbf{p})\cdot\widetilde{\mathbf{A}}(\mathbf{q})]\xi + (\xi'^{\dagger}\boldsymbol{\sigma}\xi)\cdot\widetilde{\mathbf{B}}(\mathbf{q})\Big] + \frac{e\mu}{M}\xi'^{\dagger}[\boldsymbol{\sigma}\cdot\widetilde{\mathbf{A}}(\mathbf{q})]\xi\Big\}, \end{aligned} \tag{3.2.16}$$

where it was defined the non-relativistic scattering amplitude $\mathcal{M}_{nr} = \mathcal{M}/2M$, with the normalization concerning to the effective mass M and not to the ordinary one, m.

Recovering the Born approximation, this scattering amplitude corresponds to a scattering potential as

$$\begin{aligned} V(\mathbf{r}) = \Big[\frac{m}{M}(\xi'^{\dagger}\xi)\Big] e\phi(\mathbf{r}) - \Big[\frac{\mu}{M^2}\xi'^{\dagger}[(\mathbf{p}'+\mathbf{p})\cdot\boldsymbol{\sigma}]\xi\Big] e\phi(\mathbf{r}) \\ -\frac{em}{2M^2}\Big[\xi'^{\dagger}[(\mathbf{p}'+\mathbf{p})\cdot\mathbf{A}(\mathbf{r})]\xi + (\xi'^{\dagger}\boldsymbol{\sigma}\xi)\cdot\mathbf{B}(\mathbf{r})\Big] + \frac{e\mu}{M}\xi'^{\dagger}[\boldsymbol{\sigma}\cdot\mathbf{A}(\mathbf{r})]\xi. \end{aligned} \tag{3.2.17}$$

Looking at the rightmost terms in the first and second lines on the right side of (3.2.17), we can quickly realize that this scattering potential has two new couplings compared to the usual QED potential given in (3.1.22). And looking at the leftmost terms in the first and second lines on the right side of (3.2.17), one can be seen that the “old” couplings of the usual QED receives contributions from the axial mass.

To clarify the physics meaning of these changes, let us restrict the potential (3.2.17) under the following considerations: i) Constant magnetic fields for which stands out $\mathbf{A} = \frac{1}{2}(\mathbf{B}\times\mathbf{r})$; (ii) Low-energy regime where the transferred momentum goes to zero, $\mathbf{q}\to 0$, which implies $\mathbf{p}' = \mathbf{p}$; and (iii) Initial and final spin polarizations are the same. Therefore, under these considerations and by adding up the kinetic energy in the scattering potential, we arrive out to the Pauli-Schrödinger Hamiltonian for this model:

$$\begin{aligned} H_{\text{P-S}} &= \frac{p^2}{2M^2} + V, \\ V &= \Big(\frac{me}{M}\Big)\phi - i\bigg(\frac{2e\mu}{M^2}\bigg)\langle\mathbf{S}\rangle\cdot\mathbf{E} - \Big(\frac{em}{2M^2}\Big)(\langle\mathbf{L}\rangle + 2\langle\mathbf{S}\rangle)\cdot\mathbf{B} - \bigg(-\frac{4\mu}{g_S}\bigg)\langle\mathbf{T}\rangle\cdot\mathbf{B}, \end{aligned} \tag{3.2.18}$$

where $\langle\mathbf{L}\rangle$ and $\langle\mathbf{S}\rangle$ are the same given in (3.1.24), $\mathbf{E} = -\boldsymbol{\nabla}\phi$ is the usual electric field, and $\mathbf{T}$ is the toroidal moment defined as (DUBOVIK; TUGUSHEV, 1990; EDERER; SPALDIN, 2007)

$$\mathbf{T} = \frac{1}{2}\mathbf{r}\times\mathbf{m}, \quad \mathbf{m} = g_S\frac{e}{2M}\mathbf{S}, \tag{3.2.19}$$

in which $\mathbf{m}$ is the magnetic moment.

The toroidal moment in the classical framework can be illustrated by a solenoid that is bent into a torus so that the current induces a circular magnetic field inside the solenoid, giving rise then to a toroidal moment perpendicular to the magnetic field. It violates both $\mathcal{P}$ and $\mathcal{T}$ symmetries.

The toroidal moment shows itself at the third order in multipole expansion of the vector potential expansion and it is strongly related to the non-diagonal components of the magneto-electric tensor present at the quadratic order expansion of the free energy (SPALDIN; FIEBIG; MOSTOVOY, 2008).

Let now us to make some remarks about the equation (3.2.18). First we note the presence of two new couplings due to non-Hermitian effects: (i) A coupling related to the electric dipole moment interaction $\mathbf{S} \cdot \mathbf{E}$, and (ii) A coupling related to the toroidal moment interaction $\mathbf{T} \cdot \mathbf{B}$. On the other hand, we see that the electrical and magnetic couplings receive corrections from the axial mass.

Finally, we can read the following couplings[7] in equation (3.2.18):

$$g_E^{\text{AM-QED}} = \frac{em}{M}, \quad g_{ED} = \frac{2e\mu}{M^2}, \qquad g_B^{\text{AM-QED}} = \frac{em}{2M^2}, \qquad g_T^{\text{AM-QED}} = -\frac{4\mu}{g_S}, \tag{3.2.20}$$

where g_{ED} stands for the coupling of the electric dipole moment interaction $\mathbf{S} \cdot \mathbf{E}$, g_T stands for the coupling of the toroidal moment interaction $\mathbf{T} \cdot \mathbf{B}$, and g_S is the spin g-factor.

In order to establish a comparison with the usual QED, we can expand the effective mass M and analyze the corrections given by the axial mass μ with respect to the ordinary mass m. This can be done by just making a Taylor expansion of M^{-1} and M^{-2} when $\mu \ll m$ to $\mathcal{O}(m^{-2})$. The results are summarized in table 3.4. The angular momentum and spin g-factors were obtained from the magnetic coupling g_B; we remove the $e/2m$ ratio and multiply each g-factor by their respective values accompanied by the expected values in (4.1.41), that is, 1 and 2.

After this discussion, there are some remarks worth making thanks to the axial mass: (i) It gives corrections to the couplings of the usual QED, (ii) It brings new couplings to the interaction, and (iii) The approach via non-relativistic limit from the Dirac equation can not trace back the contribution due the axial mass to the coupling, which allow us to say that the

[7] The signs of the electric dipole and toroidal couplings are defined as being the opposite as that of the interaction term between electric field and spin and between magnetic field and toroidal moment, respectively.

Table 3.4 – Couplings from the QED with fermionic axial mass.

Couplings	Values
Electric	$g_E^{\text{AM-QED}} \approx e\left(1+\frac{\mu^2}{2m^2}\right)$
Electric dipole	$g_{ED}^{\text{AM-QED}} \approx \frac{2e\mu}{m^2}\left(1+\frac{\mu^2}{m^2}\right)$
Magnetic	$g_B^{\text{AM-QED}} \approx \frac{e}{2m}\left(1+\frac{\mu^2}{m^2}\right)$
Angular momentum	$g_L^{\text{AM-QED}} \approx 1+\frac{\mu^2}{m^2}$
Spin	$g_S^{\text{AM-QED}} \approx 2+\frac{2\mu^2}{m^2}$
Toroidal	$g_T^{\text{AM-QED}} \approx -2\mu\left(1-\frac{\mu^2}{m^2}\right)$

Source: Authors (2024).

axial mass distinguishes the non-relativistic approach to tree-level QED from the non-relativistic approach to the Dirac equation, where the former carries more information about the interaction.

What explains the third remark mentioned before is the spinor normalization factor present in the calculation of the Born approximation that contains the axial mass contribution [cf. the equation (3.2.14)], which missing in the non-relativistic limit of the Dirac equation [cf. the equation (3.1.3)].

Moreover, the usual QED couplings can be recovered by just taking $\mu \to 0$ in all couplings given in the table 3.4.

However, when one neglects terms of $\mathcal{O}(m^{-2})$, the usual QED couplings are almost all recovered, except the new toroidal coupling, which becomes $g_T^{\text{AM-QED}} = -2\mu$ and does not go to zero. This consolidates that the axial mass gives non-vanishing contribution to the Pauli-Schrödinger Hamiltonian even at the leading order expansion of the effective mass.

3.3 Pauli-Schrödinger Hamiltonian in QED with fermionic axial mass and (V-A) interaction

Now we consider our non-Hermitian model not only with the axial mass, but also with an axial coupling. We will follow the non-relativistic limit of the Dirac equation and also of the tree-level QED to get the Pauli-Schrödinger Hamiltonian.

3.3.1 First approach: canonical Dirac equation with fermionic axial mass and (V-A) interaction

Now we introduce the interaction between the fermionic and electromagnetic fields in terms of a modified minimal coupling[8]:

$$p^{\alpha} \to p^{\alpha} - (g_v + g_a\gamma_5)A^{\alpha}, \tag{3.3.1}$$

where g_v represents the vector interaction and g_a, the axial interaction; the Hermitian limit is recovered by taking $g_v \to e$ and $g_a \to 0$.

We then add this modified minimal coupling into canonical Dirac equation with axial mass, which results

$$E\psi(x) = \{\boldsymbol{\alpha} \cdot [\mathbf{p} - (g_v + g_a\gamma_5)\mathbf{A}] + (g_v + g_a\gamma_5)\phi + \beta m + \beta\gamma_5\mu)\}\psi(x). \tag{3.3.2}$$

Right away, we split the fermionic field in the two-spinor representation as in (3.1.3) and use the Dirac representation for the matrices $\boldsymbol{\alpha}$, β and γ_5 to get

$$\begin{cases} E\varphi = [g_v\phi - g_a(\boldsymbol{\sigma} \cdot \mathbf{A}) + m]\varphi + [g_a\phi + \boldsymbol{\sigma} \cdot (\mathbf{p} - g_v\mathbf{A}) + \mu]\chi, \\ E\chi = [g_a\phi + \boldsymbol{\sigma} \cdot (\mathbf{p} - g_v\mathbf{A}) - \mu]\varphi + [g_v\phi - g_a(\boldsymbol{\sigma} \cdot \mathbf{A}) - m]\chi. \end{cases} \tag{3.3.3}$$

By solving the second equation from (3.3.3) for χ and using this result in the first one we get

$$E\varphi = \left\{ g_v\phi - g_a(\boldsymbol{\sigma} \cdot \mathbf{A}) + m + \frac{2g_a\phi(\boldsymbol{\sigma} \cdot \mathbf{p}) + [\boldsymbol{\sigma} \cdot (\mathbf{p} - g_v\mathbf{A})]^2 - \mu^2}{E + m - g_v\phi + g_a(\boldsymbol{\sigma} \cdot \mathbf{A})} \right\}\varphi, \tag{3.3.4}$$

where terms proportional to $g_a g_v$ and g_a^2 were neglected.

We can now consider the non-relativistic limit and assume that both interaction $g_v\phi$ and $g_a(\boldsymbol{\sigma} \cdot \mathbf{A})$ have at most a magnitude in the range of the kinetic energy, which makes both of them negligible when compared to the rest energy M. On the other hand, the total relativistic energy is approximately $E \approx M$. Taking these approximation into account, we get

$$E + m - g_v\phi + g_a(\boldsymbol{\sigma} \cdot \mathbf{A}) \approx 2m. \tag{3.3.5}$$

in which terms as $\mathcal{O}(m^{-2})$ were neglected.

Finnaly, using the approximation (3.3.5) and the property $[\boldsymbol{\sigma} \cdot (\mathbf{p} - g_v\mathbf{A})]^2 = (\mathbf{p} - g_v\mathbf{A})^2 -$

[8] This minimal coupling is related to the gauge invariance of the non-Hermitian QED under to the following combined vector and axial gauge transformation: $A_{\alpha} \to A_{\alpha} - \partial_{\alpha}\phi$, $\psi \to e^{i(g_v + g_a\gamma_5)}\psi$, and $\bar{\psi} \to \bar{\psi}e^{-i(g_v - g_a\gamma_5)}\phi$, cf. Alexandre, Bender, and Millington (2015, 2017).

$g_v(\boldsymbol{\sigma}\cdot\mathbf{B})$ into (3.3.4), we are able to obtain the Pauli-Schrödinger Hamiltonian for this model:

$$G_M\varphi = H_{\text{P-S}}^{\text{AM\&VA-Dirac}}\varphi,$$
$$H_{\text{P-S}}^{\text{AM\&(V-A)-Dirac}} = g_v\phi + \left[\frac{g_a}{m}(\boldsymbol{\sigma}\cdot\mathbf{p})\right]\phi + \frac{(\mathbf{p}-g_v\mathbf{A})^2}{2m} - \frac{g_v}{2m}(\boldsymbol{\sigma}\cdot\mathbf{B}) - g_a(\boldsymbol{\sigma}\cdot\mathbf{A}). \quad (3.3.6)$$

If we restrict our attention to constant magnetic fields, and neglect terms proportional g_v^2, this Hamiltonian reduces to

$$H_{\text{P-S}}^{\text{AM\&(V-A)-Dirac}} = \frac{\mathbf{p}^2}{2m} + g_v\phi - i\left(-\frac{2g_a}{m}\right)(\mathbf{S}\cdot\mathbf{E}) - \left(\frac{g_v}{2m}\right)(\mathbf{L}+2\mathbf{S})\cdot\mathbf{B} - \left(\frac{4m}{g_S}\frac{g_a}{g_v}\right)(\mathbf{T}\cdot\mathbf{B}), \quad (3.3.7)$$

where $G_M \approx m-\mu^2/2m$ is the non-relativistic energy given in (3.2.7), and the toroidal moment is given by

$$\mathbf{T} = \frac{1}{2}(\mathbf{r}\times\mathbf{m}), \quad \mathbf{m} = g_S\frac{g_v}{2M}\mathbf{S}, \quad (3.3.8)$$

in which $\mathbf{m}$ is the magnetic moment.

Hence, we see that the term g_a related to the axial interaction yields corrections to the electric dipole and toroidal couplings, while it does not gives any contribution to the magnetic and electric couplings.

Furthermore, the term g_a brings a new coupling when compared to the usual and axial Dirac theory, namely, the coupling to the toroidal moment interaction, which already appears in the non-relativistic limit of the three-level QED with axial mass.

Moreover, we note that at the Hermitian limit $g_a \to 0$ and $g_v \to e$ the usual $H_{\text{P-S}}$ given in equations (3.1.7), (3.2.8) and (3.1.23) is recovered.

We also observe that in this approach the same thing happens as in the non-relativistic approach to the Dirac equation with only axial mass: There is no contribution due to the axial mass to the couplings.

Finally, the table 3.5 summarize the couplings associated with the Pauli-Schrödinger Hamiltonian for the Dirac model with axial mass and (V-A) interaction.

3.3.2 Second approach: tree-level QED with fermionic axial mass and (V-A) interaction

Now we analyze the scattering of a fermion with ordinary mass m and axial mass μ by a classical electromagnetic field via (V-A) interaction. The procedure is the same as in the sections 3.1.2 and 3.2.2, distinguishing in the Lagrangian density that now turns out to be

$$\mathcal{L} = \bar{\psi}(i\gamma^\alpha\partial_\alpha - m - \mu\gamma_5)\psi - \bar{\psi}\gamma^\alpha(g_v + g_a\gamma_5)\psi A_\alpha. \quad (3.3.9)$$

Table 3.5 – Couplings from the Dirac equation with fermionic axial mass and (V-A) interaction.

Couplings	Values
Electric	$g_E^{\text{AM\&(V-A)-Dirac}} = g_v$
Electric dipole	$g_{ED}^{\text{AM\&(V-A)-Dirac}} = -\frac{2g_a}{m}$
Magnetic	$g_B^{\text{AM\&(V-A)-Dirac}} = \frac{g_v}{2m}$
Angular momentum	$g_L^{\text{AM\&(V-A)-Dirac}} = 1$
Spin	$g_S^{\text{AM\&(V-A)-Dirac}} = 2$
Toroidal	$g_T^{\text{AM\&(V-A)-Dirac}} = \frac{2mg_a}{g_v}$

Source: Authors (2024).

By observing this Lagrangian density we can see that the interaction Hamiltonian is given by

$$H_I = \int \mathrm{d}^3x\, \bar{\psi}\gamma^\alpha(g_v + g_a\gamma_5)\psi A_\alpha. \tag{3.3.10}$$

Thus the scattering amplitude for time-fixed fields will be given by

$$i\mathcal{M} = -i\bar{u}(p')\gamma^\alpha(g_v + g_a\gamma_5)u(p)\widetilde{A}_\alpha(\mathbf{q}), \tag{3.3.11}$$

where $\mathbf{q} = \mathbf{p}' - \mathbf{p}$ is the transferred momentum, with $\mathbf{p}$ and $\mathbf{p}'$ being the initial and final momenta, respectively.

Now we take the non-relativistic limit of the spinor $u(p)$ by expanding its solution and neglecting terms of $\mathcal{O}(M^{-2})$, as shown in (3.2.14). After straightforward calculations, and by neglecting terms of $\mathcal{O}(M^{-2})$, one obtains

$$\bar{u}(p')\gamma^0 u(p) = 2M\xi'^\dagger\left[\frac{m}{M} - \frac{\mu}{2M^2}(\mathbf{p}+\mathbf{p}')\cdot\boldsymbol{\sigma}\right]\xi, \tag{3.3.12a}$$

$$\bar{u}(p')\gamma^0\gamma_5 u(p) = 2M\xi'^\dagger\left[-\frac{\mu}{M} + \frac{m}{2M^2}(\mathbf{p}+\mathbf{p}')\cdot\boldsymbol{\sigma}\right]\xi, \tag{3.3.12b}$$

$$\bar{u}(p')\gamma^k u(p) = 2M\xi'^\dagger\left\{\frac{m\left[(p'+p)^k - i\epsilon^{ijk}q^i\sigma^j\right]}{2M^2} - \frac{\mu\sigma^k}{M}\right\}\xi, \tag{3.3.12c}$$

$$\bar{u}(p')\gamma^k\gamma_5 u(p) = 2M\xi'^\dagger\left\{-\frac{\mu[(p'+p)^k - i\epsilon^{ijk}q^i\sigma^j]}{2M^2} + \frac{m\sigma^k}{M}\right\}\xi. \tag{3.3.12d}$$

Moreover, we then use (3.3.12) in (3.3.11), normalize the scattering amplitude with the normalization factor $2M$ and use the Born approximation. Taking into account all these manipulations, the result is the following scattering potential:

$$
\begin{aligned}
V(\mathbf{r}) = {} & \xi'^{\dagger}\left[\frac{mg_v}{M}\left(1-\frac{\mu g_a}{mg_v}\right)\right]\xi\phi(\mathbf{r}) - \xi'^{\dagger}\left[\frac{\mu g_v}{2M^2}\left(1-\frac{mg_a}{\mu g_v}\right)(\mathbf{p}'+\mathbf{p})\cdot\boldsymbol{\sigma}\right]\xi\phi(\mathbf{r}) \\
& - \frac{mg_v}{2M^2}\left(1-\frac{\mu g_a}{mg_v}\right)\xi'^{\dagger}[(\mathbf{p}'+\mathbf{p})\cdot\mathbf{A}(\mathbf{r})+\boldsymbol{\sigma}\cdot\mathbf{B}]\xi + \frac{\mu g_v}{M}\left(1-\frac{mg_a}{\mu g_v}\right)\xi'^{\dagger}[\boldsymbol{\sigma}\cdot\mathbf{A}(\mathbf{r})]\xi.
\end{aligned}
\tag{3.3.13}
$$

Finally, we restrict our analysis to the case of constant magnetic fields, enforce the low-energy regime in such way that $\mathbf{q} \to 0$, and use the condition ${\xi'}^{\dagger}\xi = 1$ (same initial and final spin polarization), so that this scattering potential becomes

$$
\begin{aligned}
V = {} & \left[\frac{mg_v}{M}\left(1-\frac{\mu g_a}{mg_v}\right)\right]\phi - \left[\frac{2\mu g_v}{M^2}\left(1-\frac{mg_a}{\mu g_v}\right)\right]\langle\mathbf{S}\rangle\cdot\mathbf{E} \\
& - \frac{mg_v}{2M^2}\left(1-\frac{\mu g_a}{mg_v}\right)(\langle\mathbf{L}\rangle + 2\langle\mathbf{S}\rangle)\cdot\mathbf{B} - \left[-\frac{4\mu}{g_S}\left(1-\frac{mg_a}{\mu g_v}\right)\right]\langle\mathbf{T}\rangle\cdot\mathbf{B}.
\end{aligned}
\tag{3.3.14}
$$

Hence, it can be seen that the axial coupling does not induces new couplings when compared with equation (3.2.18). However, it changes their magnitudes. Moreover, we obtain the following couplings:

$$
\begin{aligned}
g_E^{\text{AM\&(V-A)-QED}} &= \frac{mg_v}{M}\left(1-\frac{\mu g_a}{mg_v}\right) \\
g_{ED}^{\text{AM\&(V-A)-QED}} &= \frac{2\mu g_v}{M^2}\left(1-\frac{mg_a}{\mu g_v}\right) \\
g_B^{\text{AM\&(V-A)-QED}} &= \frac{mg_v}{2M^2}\left(1-\frac{\mu g_a}{mg_v}\right), \\
g_T^{\text{AM\&(V-A)-QED}} &= -\frac{4\mu}{g_S}\left(1-\frac{mg_a}{\mu g_v}\right).
\end{aligned}
\tag{3.3.15}
$$

It can be noted in this approach both axial mass and axial coupling give contributions to the three couplings: electric, magnetic and toroidal.

To compare with the usual QED, we can expand the effective mass M and analyze the corrections given by the axial mass μ with respect to the ordinary mass m. This can be done by just making an Taylor expansion of M^{-1} and M^{-2} for $\mu \ll m$ to $\mathcal{O}(m^{-2})$. The results are summarized in the table 3.6.

When one takes $g_v \to e$ and $g_a \to 0$, the couplings in table 3.6 reduce to the couplings of the QED with only fermionic axial mass, as can be seen in table 3.4. And, naturally, when one takes $g_v \to e$ and $\mu, g_a \to 0$, these couplings reduce to that one of the usual QED given in table 3.2.

On the other hand, we can observe a distinction between the approach between the non-relativistic limit from the Dirac model and the non-relativistic limit from tree-level QED:

Table 3.6 – Couplings from the QED with fermionic axial mass and (V-A) interaction.

Couplings	Values
Electric	$g_E^{\text{AM\&(V-A)-QED}} \approx g_v\left(1 - \frac{\mu g_a}{m g_v} + \frac{\mu^2}{2m^2}\right)$
Electric dipole	$g_{ED}^{\text{AM\&(V-A)-QED}} \approx \frac{2g_v\mu}{m^2}\left[1 + \frac{\mu^2}{m^2} - \frac{g_a}{g_v}\left(\frac{m}{\mu} + \frac{\mu}{m}\right)\right]$
Magnetic	$g_B^{\text{AM\&(V-A)-QED}} \approx \frac{g_v}{2m}\left(1 + \frac{\mu^2}{m^2} - \frac{\mu g_a}{m g_v}\right)$
Angular momentum	$g_L^{\text{AM\&(V-A)-QED}} \approx 1 + \frac{\mu^2}{m^2} - \frac{\mu g_a}{m g_v}$
Spin	$g_S^{\text{AM\&(V-A)-QED}} \approx 2 + \frac{2\mu^2}{m^2} - \frac{2\mu g_a}{m g_v}$
Toroidal	$g_T^{\text{AM\&(V-A)-QED}} \approx -2\mu\left[1 - \frac{\mu^2}{m^2} - \frac{g_a}{g_v}\left(\frac{m}{\mu} - \frac{2\mu}{m}\right)\right]$

Source: Authors (2024).

The former can not trace the axial mass contributions to the couplings [compare the tables 3.5 and 3.6].

Furthermore, when we consider the vanishing axial mass limit, $\mu \to 0$, one recovers exactly the results in table 3.5, which reinforce the fact that the non-relativistic limit from the Dirac theory does not receive contributions induced by the axial mass.

We reaffirm why this happens: The spinor normalization factor present in the calculation of the non-relativistic limit of the tree-level QED receives contribution from axial mass [cf. the equation (3.2.14)], whereas in the non-relativistic limit of the Dirac equation it does not [cf. the equation (3.1.3)].

3.4 Some remarks on the couplings

It was seen that, when compared with the usual non-relativistic tree-level QED, the QED with fermionic axial mass and (V-A) interaction has new contributions to the electric coupling, and another one to the magnetic coupling, which ends up contributing to the angular and spin g-factors. On the other hand, it introduces two new couplings: One related to the electric dipole moment interaction and other to the toroidal moment interaction.

Here we will treat each coupling separately: that one from to the electric dipole moment interaction, te corrected spin g-factor, and that one from the toroidal moment interaction.

3.4.1 g_{ED} coupling

According to the table 3.6, the coupling associated to the electric dipole interaction in the Schrödinger Hamiltonian of the non-Hermitian QED is

$$g_{ED}^{\text{AM\&(V-A)-QED}} \approx \frac{2g_v\mu}{m^2}\left[1+\frac{\mu^2}{m^2}-\frac{g_a}{g_v}\left(\frac{m}{\mu}+\frac{\mu}{m}\right)\right]. \tag{3.4.16}$$

On the other hand, Baron et al. (1989) states that electric dipole moment of the electron is restrict to the following inequality:

$$d_e \leq 8.7\times 10^{-29} g_v\,\text{cm} = 7\times 10^{-25} g_v\,\text{eV}^{-1} = 3.5\times 10^{-19}\cdot\frac{g_v}{m}. \tag{3.4.17}$$

Gathering the results in equations (3.4.16) and (3.4.17) we conclude that

$$\frac{2\mu}{m}\left[1+\frac{\mu^2}{m^2}-\frac{g_a}{g_v}\left(\frac{m}{\mu}+\frac{\mu}{m_e}\right)\right] \leq 3.5\times 10^{-19}, \tag{3.4.18}$$

and then find a relation between our parameters g_a and μ.

3.4.2 g_S coupling

According to the table 3.6, the correction to the spin g-faction given by the non-Hermitian QED is

$$g_S^{\text{AM\&(V-A)-QED}} \approx 2+\frac{2\mu^2}{m^2}-\frac{2\mu g_a}{m g_v} \tag{3.4.19}$$

so that the anomalous magnetic moment is given by

$$a^{\text{AM\&(V-A)-QED}} = \frac{1}{2}g_S^{\text{AM\&(V-A)-QED}} - 1 = \frac{\mu^2}{m^2}-\frac{\mu g_a}{m g_v}. \tag{3.4.20}$$

On the other hand, Fan et al. (2023) presents the measurement of the anomalous magnetic moment of the electron:

$$a_e = 0.001\,159\,652\,180\,59\,(13), \tag{3.4.21}$$

so that the associated error is

$$\delta a_e = 13\times 10^{-14}. \tag{3.4.22}$$

We thus conclude that the contribution of the non-Hermitian QED to the anomalous magnetic moment of the electron is less than or equal to the experimental error in (3.4.22):

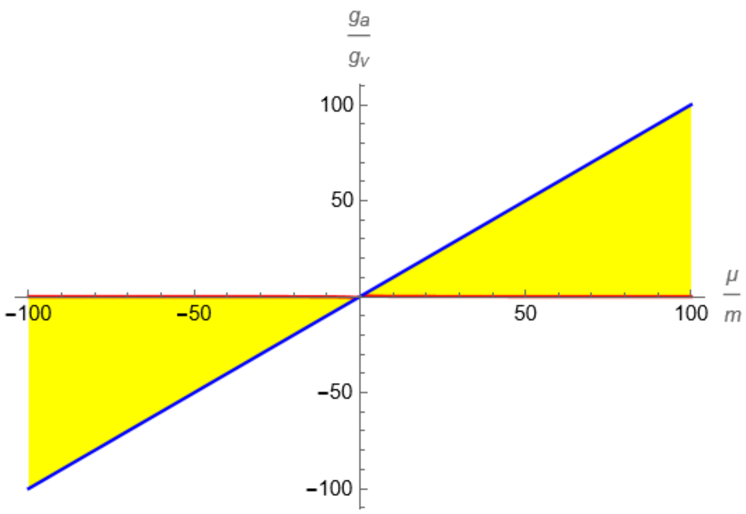
$$\frac{\mu^2}{m^2}-\frac{\mu g_a}{m g_v} \leq 13\times 10^{-14}. \tag{3.4.23}$$

Gathering equations (3.4.18) and (3.4.23) we have a system of equations, which can be solve to give the restriction to the axial parameters μ, g_a. Solving it:

$$\begin{cases} \left[\dfrac{-1+\dfrac{\mu}{m}\cdot 5.7\times 10^{18}}{\left(1+\dfrac{\mu^2}{m^2}\right)\cdot 5.7\times 10^{18}}\right] g_v \leq g_a \leq \left(\dfrac{\mu}{m} - 13\times 10^{-14}\cdot\dfrac{m}{\mu}\right) g_v \quad \text{if} \quad \mu < 0, \\ \left[\dfrac{-1+\dfrac{\mu}{m}\cdot 5.7\times 10^{18}}{\left(1+\dfrac{\mu^2}{m^2}\right)\cdot 5.7\times 10^{18}}\right] g_v \leq g_a \quad \text{if} \quad \mu > 0. \end{cases} \tag{3.4.24}$$

The graph in figure 3.2 shown the behavior of the g_a/g_v ratio in terms of μ/m according to (3.5.45). It is possible to see that the greater the μ/m ratio, the g_a/g_v ratio can assume greater values, meaning that $g_a > g_v$; and that the smaller the μ/m ratio, the g_a/g_v can assume smaller values, meaning that $g_a < g_v$.

Figure 3.2 – Behavior of the g_a/g_v ratio in terms of μ/m according to first equation of (3.4.24). In the yellow region are the accessible values of g_a/g_v as a function of m/μ.



Source: Authors (2024).

3.4.3 g_T coupling

Toroidal moments was extensively explored in classical electrodynamics (DUBOVIK; TUGUSHEV, 1990), solid-state physics (SPALDIN; FIEBIG; MOSTOVOY, 2008; EDERER; SPALDIN, 2007) and particle physics (CABRAL-ROSETTI; MORENO; ROSADO, 2002; CABRAL-ROSETTI; MONDRAGÓN; PÉREZ, 2009; BUKINA; DUBOVIK; KUZNETSOV, 1998).

Particularly, the toroidal moment has attracted attention in particle physics thanks to the possibility of the neutrino to be Majorana-like. Then it would not have electric and magnetic dipole moment but only the toroidal (KAYSER, 1982) or the anapole one (BOUDJEMA et al., 1989).

Is is well-known that the electronic neutrino with a toroidal moment produces a transition radiation when crossing the interface between two different media and its toroidal moment, which is given by $\tau_{\nu_e} = -g_v \mathcal{T}(0)/m_{\nu_e}^3$, where $\mathcal{T}(0)$ corresponds to the toroidal form factor (BUKINA; DUBOVIK; KUZNETSOV, 1998); the electric charge g_v was assumed to be negative and the neutrino mass m_{ν_e} was added to correct the dimensionality for our purposes.

Let us consider this observation in our model in order to assess the parameter μ. The toroidal form factor reduces to $g_T^{\text{AM\&(V-A)-QED}} = -2\mu$, where was neglected all terms of $\mathcal{O}(m^{-1})$ [see table 3.6]. We can then relate the electronic neutrino axial mass to its toroidal moment as

$$\mu_{\nu_e} \sim \frac{\tau_{\nu_e} m_{\nu_e}^3}{2 g_v} \approx 10^{-25}\,\mathrm{eV}, \tag{3.4.25}$$

where it was assumed that the theoretical prediction $\tau_{\nu_e} \sim g_v 10^{-34}\,\mathrm{cm}^2$ exposed by Bukina, Dubovik, and Kuznetsov (1998) is valid and also that $m_{\nu_e} \sim 1\,\mathrm{eV}$. These results respects the condition $\mu_{\nu_e} \ll m_{\nu_e}$.

It is worth mentioning that the toroidal moment of the Dirac neutrino is just half of the Majorana one in the massless limit and, moreover, when the initial and final states are the same the toroidal moment reduces to the anapole moment. Under these circumstances it is possible to calculate the one-loop correction for toroidal moment of the neutrino in the electroweak theory. The toroidal moment of the neutrino at the one-loop correction has an order of $10^{-35}\,\mathrm{cm}^2$ (CABRAL-ROSETTI; MORENO; ROSADO, 2002; CABRAL-ROSETTI; MONDRAGÓN; PÉREZ, 2009).

3.5 Axial mass and axial coupling contributions to the hydrogen atom

In order to give some application of our results, we applied the hydrogen atom potential to our Pauli-Schrödinger Hamiltonian obtained from the non-relativistic regime for the non-hermitian QED with both axial and vector couplings, which is given in the equation (3.3.14).

However, looking for some simplicity, we dispense the contribution of the magnetic field and work on only one possibilities for the electric potential: (i) without spin contribution.

We then will find out the hydrogen energy spectrum for the potential given in (3.3.14). For this case one sets the magnetic field as null ($\mathbf{B} = 0$) take spinless case ($\mathbf{S} = 0$), and assumes that the electric potential is Coulombian: $\phi = -Zg_v/r$. Then we obtain the following Hamiltonian:

$$H = \frac{p^2}{2M} - \frac{m}{M}\left(1 - \frac{g_a}{g_v}\frac{\mu}{m}\right)\frac{Zg_v^2}{r}. \tag{3.5.26}$$

The Schrödinger equation for this Hamiltonian is $i\,\partial/\partial t\,\psi = H\psi$. For stationary states, the solution is

$$\psi(\mathbf{r}) = \varphi(\mathbf{r})e^{-iEt}, \tag{3.5.27}$$

so that we are left with the following equation

$$E\psi(\mathbf{r}) = H\psi(\mathbf{r}) = HR(r)\Theta(\theta)\Phi(\varphi), \tag{3.5.28}$$

where we use spherical coordinates and split the solution into radial and angular parts. Making use of the above ansatz, we get the three differential equations:

$$\begin{aligned}
&\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2} = -k^2\Phi,\\
&-\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \frac{k^2}{\sin^2\theta}\Theta = \alpha\Theta,\\
&\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \left[2ME + \frac{2mZg_v^2}{r}\left(1 - \frac{g_a}{g_v}\frac{\mu}{m}\right)\right]R = \frac{\alpha}{r^2}R,
\end{aligned} \tag{3.5.29}$$

where k and α are arbitrary constants which must be evaluated in such way that the solutions are regular.

The equation for Φ has as solution an exponential function

$$\Phi(\varphi) = e^{ik\varphi}, \quad |k| = 0, 1, 2, \ldots, \tag{3.5.30}$$

where the discrete values of k are found by imposing that the solution ψ is single-valued, so that $\Phi(0) = \Phi(2\pi)$.

On the other hand, the equation for Θ, when one does $x = \cos\theta$, becomes the differential equation that has as solution the associated Legendre functions and the parameter α must satisfies (EISBERG, 1961)

$$\alpha = l(l+1), \quad l = |m|, |m|+1, |m+2|, \ldots. \tag{3.5.31}$$

Finally, one replaces the result (3.5.31) in the equation for r and make the change of variable as $\rho = 2\beta r$, where $\beta^2 = -2ME$ and get

$$\frac{1}{\rho^2}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^2\frac{\mathrm{d}R}{\mathrm{d}\rho}\right) + \left[-\frac{1}{4} - \frac{l(l+1)}{\rho^2} + \frac{\gamma}{\rho}\right] = 0, \tag{3.5.32}$$

where

$$\gamma = \frac{mZg_v^2}{\beta}\left(1 - \frac{g_a}{g_v}\frac{\mu}{m}\right). \tag{3.5.33}$$

The solutions of the equation (3.5.32) are the associated Laguerre functions and it is regular only for (EISBERG, 1961)

$$\gamma = n, \quad n = l+1, l+2, \ldots. \tag{3.5.34}$$

We note that the equations (3.5.30), (3.5.31) and (3.5.34) establish the restriction conditions to our quantum numbers n, l and k. Hence, it is convenient to express these conditions in the following way:

$$\begin{aligned} n &= 1, 2, 3, \ldots, \\ l &= 0, 1, 2, \ldots, n-1, \\ m &= -l, -l+1, \ldots, 0, \ldots, +l-1, +l. \end{aligned} \tag{3.5.35}$$

Finally, considering these observations, we are capable to express our energy in terms of the quantum number n through the relation $\beta^2 = -2ME$ and the (3.5.33). The result is

$$E_n = -\left(1 - \frac{g_a}{g_v}\frac{\mu}{m}\right)^2 \frac{m^2 Z^2 g_v^4}{2Mn^2}, \tag{3.5.36}$$

where non-linear terms on the axial interaction parameter g_a were neglected.

If we expand the effective mass M to the quadratic order on the axial mass we arrive at

$$E_n = -\left[1 + \frac{\mu^2}{2m^2}\left(1 - \frac{4g_a}{g_v}\frac{m}{\mu}\right)\right]\frac{mZ^2 g_v^4}{2n^2}. \tag{3.5.37}$$

When one makes $\mu \to 0$, $g_v = e$, and $Z = 1$, one recovers the usual energy spectrum for the hydrogen atom:

$$\lim_{\mu \to 0} E_n = -\frac{me^4}{2n^2}. \tag{3.5.38}$$

On the other hand, when one makes $g_a = 0$, $g_v = e$, $Z = 1$, and allows $\mu \neq 0$, we see that still there is a deviation from the usual result by $\mu^2/2m^2$:

$$\lim_{g_a \to 0} E_n = -\left(1 + \frac{\mu^2}{2m^2}\right)\frac{me^4}{2n^2}. \tag{3.5.39}$$

Finally, we can observe in the equation (3.5.37) that the correction to the spectrum of the hydrogen atom resides on the terms insides the bracket when one sets $Z = 1$. Inasmuch, the ground-energy state, which occurs for $n = 1$, is given by

$$E_1^H = -\left[1 + \frac{\mu^2}{2m^2}\left(1 - \frac{4g_a}{g_v}\frac{m}{\mu}\right)\right]\frac{me^4}{2} = -[1 + \delta E_1^H(\mu, g_a)]\frac{mg_v^4}{2}, \tag{3.5.40}$$

where

$$\delta E_1^H(\mu, g_a) = \frac{\mu^2}{2m^2}\left(1 - \frac{4g_a}{g_v}\frac{m}{\mu}\right) \tag{3.5.41}$$

is the correction we get for the spectrum of the hydrogen atom.

On the other hand, according to the NIST Atomic Spectra Database (KRAMIDA; RALCHENKO; READER, 2023), the reported ground-energy state of the hydrogen atom is

$$(E_1^H)_{\text{Rep}} = -13.598\,434\,599\,702(12)\,\text{eV}, \tag{3.5.42}$$

so that the associated error is

$$\delta(E_1^H)_{\text{Rep}} = 1.2 \times 10^{-11}\,\text{eV}. \tag{3.5.43}$$

Thus, it was expected that the associated error to the reported value establishes a limit on our correction $\delta E_1^H(\mu, g_a)$ given in (3.5.41), that is,

$$\delta E_1^H(\mu, g_a) = \frac{\mu^2}{2m^2}\left(1 - \frac{4g_a}{g_v}\frac{m}{\mu}\right) \leq 1.2 \times 10^{-11}, \tag{3.5.44}$$

where the dimension was removed since the correction we are dealing on is dimensionless. Solving (3.5.44) for g_a, we get

$$\begin{cases} g_a \leq \left(-3 + 1.25 \times 10^{11} \cdot \dfrac{\mu^2}{m^2}\right) \cdot 2 \times 10^{-12} \cdot \dfrac{mg_v}{\mu} & \text{if} \quad \mu < 0 \text{ and } g_v > 0, \\ g_a \geq \left(-3 + 1.25 \times 10^{11} \cdot \dfrac{\mu^2}{m^2}\right) \cdot 2 \times 10^{-12} \cdot \dfrac{mg_v}{\mu} & \text{if} \quad \mu > 0 \text{ and } g_v > 0. \end{cases} \tag{3.5.45}$$

In section 3.4, we have seen that the axial mass of an electronic neutrino has approximately the value 10^{-25} eV for which the ordinary mass is approximately $1\,\text{eV}$ [cf. equation (3.4.25)]. Then for the electron with a ordinary mass given by approximately 10^6 eV, its axial mass will be order of 10^{-31} eV if we preserve the magnitude difference between the axial and ordinary mass. We, therefore, conclude that the μ/m ratio has order of 10^{-25} as for the electronic neutrino. By considering this approximation for the μ/m ratio, we obtain the following inequality for the parameter g_a:

$$\begin{cases} g_a \leq -10^{13} g_v. & \text{if} \quad \mu < 0 \text{ and } g_v > 0, \\ g_a \geq -10^{13} g_v & \text{if} \quad \mu > 0 \text{ and } g_v > 0. \end{cases} \tag{3.5.46}$$

It can be seen that if the axial mass μ is negative then the axial interaction parameter g_a can assume only very small values compared to g_v. On the other hand, if the axial mass μ is positive, the axial interaction parameter g_a cannot have a very negligible value compared to g_v.

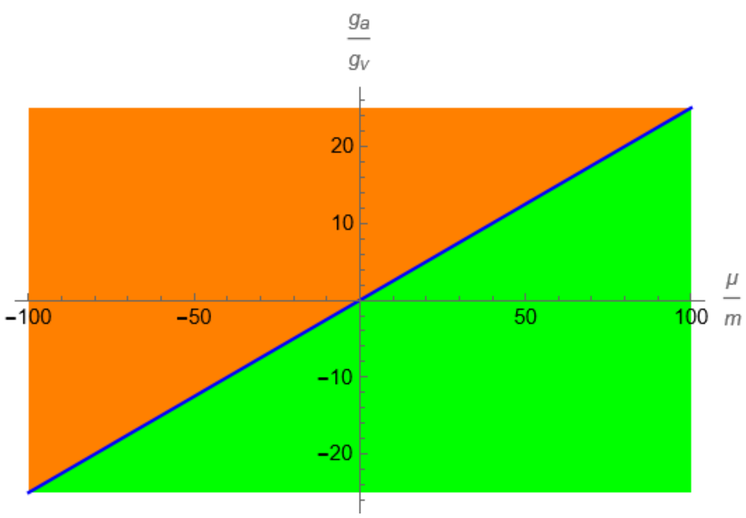
On the other hand, if we assume that the axial mass of the electron has the same order that its ordinary mass, i.e., $\mu/m = 1$, then the (3.5.45) provides

$$\begin{cases} g_a \leq 10^{-1} g_v. & \text{if} \quad \mu < 0 \text{ and } g_v > 0, \\ g_a \geq 10^{-1} g_v & \text{if} \quad \mu > 0 \text{ and } g_v > 0. \end{cases} \tag{3.5.47}$$

We thus see that when the value of the axial mass μ approximates to the ordinary mass m, the axial interaction parameter g_a can takes values close to g_v.

The graph in figure 3.3 shown the behavior of the g_a/g_v ratio in terms of μ/m according to (3.5.45). It can be seen that g_a/g_v can assume large and small values as much as one wants for both conditions $\mu > 0$ and $\mu < 0$.

Figure 3.3 – Behavior of the g_a/g_v ratio as a function of μ/m according to (3.5.45). The values of g_a/g_v below the blue line (green region) corresponds to the first equation in (3.5.45); that values above the blue line (orange region), to the second equation.



Source: Authors (2024).

4 ANOMALIES IN NON-HERMITIAN QED

The non-Hermitian QED is a recent proposed model, thus it must be explored in several aspects. From this perspective, in this chapter we will investigate some typical anomalies of QED-like models in (1+1) dimensional spacetime, namely, the Schwinger-like models. In our case, we have the Chiral Schwinger model with non-unitary couplings.

The section 4.1 is intended to find out the photon mass by looking to the pole of momentum in the propagator corrected by the one loop contribution to the vacuum polarization tensor.

The section 4.2, on the other hand, deals specifically with the chiral anomaly, which is calculated non-perturbatively by using the Fujikawa method.

4.1 Chiral Schwinger model with non-unitary couplings

The Schwinger model describes the QED in (1+1) dimensions,[1] whose Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \bar{\psi}(i\gamma^{\alpha}\partial_{\alpha} - m)\psi - e\bar{\psi}\gamma^{\alpha}\psi A_{\alpha}. \tag{4.1.1}$$

One of its remarkable characteristics is that the photon dynamically acquires mass while preserving the gauge invariance.

We can ask ourselves the following: Can Schwinger mass phenomena happen in modified QED-like models? We examine this question for the the Chiral Schwinger model with non-unitary couplings, whose Lagrangian density is

$$\mathcal{L} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \bar{\psi}(i\gamma^{\alpha}\partial_{\alpha} - m - \mu\gamma_5)\psi - \bar{\psi}\gamma^{\alpha}(g_v + g_a\gamma_5)\psi A_{\alpha}, \tag{4.1.2}$$

which is precisely the expression (3.3.9), considered in the non-Hermitian framework.

In order to answer this question we cast the follow observation: If the non-tensor structure of the vacuum polarization tensor is regular when the squared photon momentum goes to zero, then the photon propagator will have a pole at $p^2 = 0$, meaning this particle remains non-

[1] Another condition is that the fermions are massless, however this limit will be taken only in the final part of our analysis.

massive; If the non-tensor structure of the vacuum polarization tensor is non-regular, otherwise, the photon propagator will have a pole at $p^2 = m_\gamma^2$, and the photon acquires a mass[2] m_γ.

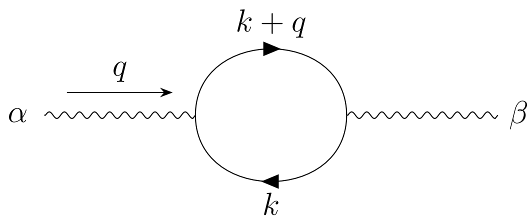
So the script to answer the question is simple. First calculate the vacuum polarization tensor to a one-loop and then takes a limit in which the squared photon momentum goes to zero. If this limit is regular, the particle remains non-massive; on the other hand, the particle acquires mass if this limit is non-regular.

We will review first how this phenomenon works in the usual Schwinger model and then apply the analysis for the Chiral Schwinger model with non-unitary couplings.

4.1.1 Schwinger model

The Feynman diagram that originates the one-loop contribution to the vaccum polarization tensor is depicted in figure 4.1.

Figure 4.1 – One-loop contribution to the vacuum polarization tensor.



Source: Authors (2024).

To calculate the amplitude related to this diagram, we need to know how to express it mathematically. This is made in terms of the Feynman rules (GRIFFITHS, 2008, sec. 7.5), which can be derived from the Lagrangian density.

By observing the diagram, we see that besides the external two photon legs we have one loop containing two free-fermionic propagators and two interaction vertices. So, at a first glance, we need the rule of interaction between the fermionic and gauge field, the rule of the fermionic propagator, and the rule of a closed fermionic loop.

For the Lagrangian density given in equation (4.1.1) we have the following rule of interaction between the fermionic field ψ and the gauge field A_α:

$$\text{vertex in Feynman diagrams} = -ie\gamma^\alpha, \tag{4.1.3}$$

[2] There are another ways to find out the photon mass in the Schwinger model. For a path integral perspective cf. Das (2021, sec. 13.2).

This means that for every vertex in the diagram we introduce the factor given in (4.1.3).

The other needed rule is the free-fermionic propagator. To find it, we consider the fermionic free part in the Lagrangian density given in (4.1.1) and the Euler-Lagrange equations. Combining them we are leading to the well-known Dirac equation

$$(i\gamma^{\alpha}\partial_{\alpha} - m)\psi(x) = 0. \tag{4.1.4}$$

The free-fermionic propagator $S(x-y)$ is a formal solution for the equation (4.1.4) modified with a point source, i.e.,

$$(i\gamma^{\alpha}\partial_{\alpha} - m)S(x-y) = i\delta^{4}(x-y). \tag{4.1.5}$$

Moreover, the solution of the free-fermionic propagator has the simple form in momentum space

$$\widetilde{S}(p) = \lim_{\epsilon\to 0}\frac{i(\gamma^{\alpha}p_{\alpha} + m)}{p^{2} - m^{2} + i\epsilon}, \tag{4.1.6}$$

where the imaginary factor is introduced when one uses the Feynman prescription. Thus each internal fermionic line in the diagram corresponds to the propagator given in equation (4.1.6).

We now need to take into account the spin-statistic of the loop we are concerning. Since we dealing with fermions, the closed fermionic loop receives a global negative sign and a trace over all quantities composing the loop. Then we also have this rule:

$$\text{Fermionic loop in Feynman diagrams} = -\operatorname{tr}(\text{all quantities composing the loop}). \tag{4.1.7}$$

The last one rule is the simpler: After applying all previous rules one needs to integrate over all internal momenta with a factor $(2\pi)^{-2}$ for each one, and then finally multiply by the imaginary unit to get the amplitude.

By following these rules, one gets the amplitude of the one-loop contribution to the vacuum polarization tensor:

$$i\Pi_{2}^{\alpha\beta}(q) = -e^{2}\lim_{\epsilon\to 0}\int\frac{\mathrm{d}^{2}k}{(2\pi)^{2}}\frac{N^{\alpha\beta}(k,q)}{(k^{2}-m^{2}+i\epsilon)[(k+q)^{2}-m^{2}+i\epsilon]}, \tag{4.1.8}$$

where

$$N^{\alpha\beta}(k,q) = \operatorname{tr}\left\{\gamma^{\alpha}(\gamma^{\lambda}k_{\lambda} + m)\gamma^{\beta}[\gamma^{\nu}(k+q)_{\nu} + m]\right\}. \tag{4.1.9}$$

To find the correct result of this amplitude it is necessary to apply dimensional regularization. This procedure takes the quantities presents in the integral to a dimension d. Thus the equation (4.1.8) becomes

$$i\Pi_{2}^{\alpha\beta}(q) = \lim_{\epsilon\to 0}\int\frac{\mathrm{d}^{d}k}{(2\pi)^{d}}\frac{N^{\alpha\beta}(k,q)}{(k^{2}-m^{2}+i\epsilon)[(k+q)^{2}-m^{2}+i\epsilon]}. \tag{4.1.10}$$

The equation (4.1.9) preserves its structure, but with the γ-matrices and the internal momentum k being now d-dimensional quantities.

After correct calculating the amplitude we take the limit $d \to 2$ and recover the right dimension.[3]

In fact, it is possible to calculate the integral given in equation (4.1.8) without using dimensional regularization, however this breaks the gauge invariance, making the final result not gauge invariant, which is in disagreement with the well-known structure of the vacuum polarization tensor.

Now we follow a series of steps to reach the result of the integral.

4.1.1.1 Reduction by trace properties

The trace of γ-matrices and its products is well-known in the two-dimensional spacetime. However, we need to work on a d-dimensional spacetime. We would then think that these traces properties of γ-matrices change, yet they do not change.

The reason is that the tensor structure of trace properties in d-dimensional spacetime remains the same as in two-dimensional spacetime, changing only the global factor $\mathrm{tr}(1)$, which can be defined a prior as $\mathrm{tr}(1) = 2$.

Then we have the following trace properties:

$$\mathrm{tr}\big(\gamma^\alpha \gamma^\beta\big) = 2g^{\alpha\beta}, \tag{4.1.11a}$$

$$\mathrm{tr}\big(\gamma^\alpha \gamma^\lambda \gamma^\beta\big) = 0, \tag{4.1.11b}$$

$$\mathrm{tr}\big(\gamma^\alpha \gamma^\lambda \gamma^\beta \gamma^\nu\big) = 2\big(g^{\alpha\lambda} g^{\beta\nu} - g^{\alpha\beta} g^{\lambda\nu} + g^{\alpha\nu} g^{\lambda\beta}\big) \tag{4.1.11c}$$

By using equations (4.1.11a) to (4.1.11c) in the numerator $N^{\alpha\beta}(k,q)$ given in equation (4.1.9) we get

$$N^{\alpha\beta}(k,q) = 2\big\{k^\alpha (k+q)^\beta + k^\beta (k+q)^\alpha - g^{\alpha\beta}\big[k\cdot(k+q) - m^2\big]\big\}. \tag{4.1.12}$$

[3] It is worth to mentioning that if we want to make the coupling e dimensionless then it is needed to make the change $e \to \lambda^{(4-d)/2}$, with λ being a arbitrary mass parameter. However, we assume that in both usual and Chiral Schwinger models, where spacetime is (1+1), the coupling has dimension of mass. When the usual QED in (3+1) is mentioned we add the parameter λ [cf. equation (4.1.35)].

4.1.1.2 Reduction by Feynman parametrization

The Feynman parametrization for two parameters A and B in the denominator is given by

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \, \frac{1}{[Ax + (1-x)B]^2}. \tag{4.1.13}$$

If one sets $A = [(k+q)^2 - m^2 + i\epsilon]$ and $B = [(k^2 - m^2) + i\epsilon]$ in the equation (4.1.13) it is possible to get the following result:

$$\frac{1}{(k^2 - m^2 + i\epsilon)[(k+m)^2 - m^2 + i\epsilon]} = \int_0^1 \mathrm{d}x \, \frac{1}{(l^2 - \Delta_m + i\epsilon)^2}, \tag{4.1.14}$$

where

$$l = k + xq, \tag{4.1.15}$$

$$\Delta_m = m^2 - x(1-x)q^2. \tag{4.1.16}$$

Then the numerator $N^{\alpha\beta}$ given in equation (4.1.12), after the change of variable (4.1.15), becomes

$$N^{\alpha\beta}(k,q) \to N^{\alpha\beta}(l,x,q) = 2\Big(2l^\alpha l^\beta - g^{\alpha\beta} l^2 + K^{\alpha\beta}(x,q) + Q^{\alpha\beta}_\lambda(x,q) l^\lambda\Big), \tag{4.1.17}$$

where

$$K^{\alpha\beta}(x,q) = g^{\alpha\beta}[m^2 + x(1-x)q^2] - 2x(1-x)q^\alpha q^\beta, \tag{4.1.18a}$$

$$Q^{\alpha\beta}_\lambda(x,q) = (1-2x)(\delta^\alpha_\lambda q^\beta + \delta^\beta_\lambda q^\alpha - g^{\alpha\beta} q_\lambda). \tag{4.1.18b}$$

We, therefore, use the equations (4.1.14) and (4.1.17) in equation (4.1.10) so that the vacuum polarization tensor becomes

$$i\Pi_2^{\alpha\beta}(q) = -2e^2 \lim_{\epsilon\to 0} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{2l^\alpha l^\beta - g^{\alpha\beta} l^2 + K^{\alpha\beta}(x,q) + Q^{\alpha\beta}_\lambda(x,q) l^\lambda}{(l^2 - \Delta_m + i\epsilon)^2}. \tag{4.1.19}$$

4.1.1.3 Reduction by symmetry

Firstly, we can observe that the denominator of the function in the integral is even with respect to integrated variable l, since the transformation $l \to -l$ leave it unchanged.

On the other hand, the far-right term in the numerator of the function in the integral is linear with respect to integrated variable l so that the function in the integral

$$I_{l^\lambda} = \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^\lambda}{(l^2 - \Delta_m + i\epsilon)^2} \tag{4.1.20}$$

is odd as a whole.

If we split the measure $\mathrm{d}^d l$, at least one of the d measures is over the component l_λ with range $(-\infty, +\infty)$. Then we are left with an integral of an odd function over a symmetric integral, which is recognized null. Thus,

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^\lambda}{(l^2 - \Delta_m + i\epsilon)^2} = 0. \tag{4.1.21}$$

We now observe that the far-left term in the numerator of the function of the integral in equation (4.1.19) compose, with the denominator and without the factor 2, the following integral contributing to the vacuum polarization tensor:

$$I_{l^\alpha l^\beta} = \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^\alpha l^\beta}{(l^2 - \Delta_m + i\epsilon)^2}. \tag{4.1.22}$$

If $\alpha \neq \beta$, then we can split the measure $\mathrm{d}^d l$ as before and conclude that this integral is null. So the integral $I_{l^\alpha l^\beta}$ is non-vanishing only for $\alpha = \beta$. For this case, the Lorentz invariance imposes that the value of the integral should also be covariant, so that its result should be

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^\alpha l^\beta}{(l^2 - \Delta_m + i\epsilon)^2} = \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{C g^{\alpha\beta}}{(l^2 - \Delta_m + i\epsilon)^2}. \tag{4.1.23}$$

If we contract each side with $g_{\alpha\beta}$ and use the metric contraction $g_{\alpha\beta} g^{\alpha\beta} = d$ for the d-dimensional spacetime, we find $C = l^2/d$ so that the integral we are interest in reduces to

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^\alpha l^\beta}{(l^2 - \Delta_m + i\epsilon)^2} = \frac{g^{\alpha\beta}}{d} \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^2}{(l^2 - \Delta_m + i\epsilon)^2}. \tag{4.1.24}$$

We then use equations (4.1.21) and (4.1.24) in equation (4.1.19) yielding

$$\begin{aligned} i\Pi_2^{\alpha\beta}(q) = -2e^2 \lim_{\epsilon \to 0} \int_0^1 \mathrm{d}x \Bigg[\left(\frac{2}{d} - 1 \right) g^{\alpha\beta} \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^2}{(l^2 - \Delta_m + i\epsilon)^2} \\ + K^{\alpha\beta}(x, q) \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{(l^2 - \Delta_m + i\epsilon)^2} \Bigg]. \end{aligned} \tag{4.1.25}$$

4.1.1.4 Wick rotation

Formally, the vacuum polarization tensor is given in terms of an integral over the Minkowski spacetime. To evaluate it we use the complex analysis integration techniques in such way that the integral over Minkowski spacetime is replaced by an integral over the Euclidean spacetime. It is the Wick rotation that allow us to realize this procedure.

In simple terms, the Wick rotation is achieved by making a change of variable for the temporal component of the integrated momentum. For our purpose, the following change of variable will be made: $l^0 \to il^0_E$, where the subscript "E" means that this component is now part of a Euclidean vector.

What we gain from this is that the difference $l^2 - \Delta_m$ [present in the denominator in both d-dimensional integrals given in equation (4.1.25)] becomes a sum $l_E^2 + \Delta_m$, with l_E being a d-dimensional Euclidean vector, and this allows us to take the limit $\epsilon \to 0$. Then we are left with solvable d-dimensional Euclidean integrals.

In fact, it can be shown that

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{(l^2 - \Delta_m + i\epsilon)^2} = i \int \frac{\mathrm{d}^d l_E}{(2\pi)^d} \frac{1}{(l_E^2 + \Delta_m - i\epsilon)^2}, \tag{4.1.26a}$$

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^2}{(l^2 - \Delta_m + i\epsilon)^2} = -i \int \frac{\mathrm{d}^d l_E}{(2\pi)^d} \frac{l_E^2}{(l_E^2 + \Delta_m - i\epsilon)^2}. \tag{4.1.26b}$$

Hence, using the equations (4.1.26a) and (4.1.26b) in equation (4.1.25), we get

$$\begin{aligned} i\Pi_2^{\alpha\beta}(q) = -2ie^2 \int_0^1 \mathrm{d}x \Bigg[\left(1 - \frac{2}{d}\right) g^{\alpha\beta} \int \frac{\mathrm{d}^d l_E}{(2\pi)^d} \frac{l_E^2}{(l_E^2 + \Delta_m)^2} \\ + K^{\alpha\beta}(x,q) \int \frac{\mathrm{d}^d l_E}{(2\pi)^d} \frac{1}{(l_E^2 + \Delta_m)^2} \Bigg], \end{aligned} \tag{4.1.27}$$

where the limit $\epsilon \to 0$ has already been taken as a consequence of the Wick rotation.

It is worth to note that this is the main step that distinguishes the dimensional regularization from a directly calculation of the vacuum polarization tensor without regularization. Indeed, if one takes the limit $d \to 2$ in the equation (4.1.25), which will occur in the directly calculation, the first integral in this expression vanishes, and it is exactly this fact that would spoils the gauge invariance.

4.1.1.5 d-dimensional integrals and Γ-function

The two d-dimensional integrals present in equation (4.1.27) can be solved by a well-known procedure.

Firstly, split the measure $\mathrm{d}^d l_E$ in angular and radial components. The angular contribution is just the d-dimensional solid angle, which can be written in terms of the Γ-function via manipulation of the Gaussian integral:

$$\int \mathrm{d}\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{4.1.28}$$

The radial contribution can be rearranged until it reaches the form of a B-function which, in turn, can be written in terms of the Γ-function (PESKIN; SCHROEDER, 1995, p. 249 and 250).

The results are

$$\int \frac{\mathrm{d}^d l_E}{(2\pi)^d} \frac{1}{(l_E^2 + \Delta_m)^2} = \frac{1}{(4\pi)^{d/2}} \Gamma\left(2 - \frac{d}{2}\right) \frac{1}{\Delta_m^{2-d/2}}, \tag{4.1.29a}$$

$$\int \frac{\mathrm{d}^d l_E}{(2\pi)^d} \frac{l_E^2}{(l_E^2 + \Delta_m)^2} = \frac{1}{(4\pi)^{d/2}} \left(\frac{d}{2}\right) \Gamma\left(1 - \frac{d}{2}\right) \frac{1}{\Delta_m^{1-d/2}}. \tag{4.1.29b}$$

We observe that the equation (4.1.29b) has a pole $\Gamma(0)$ for $d \to 2$. However, this pole is fictitious, because it is removed in the vacuum polarization tensor.

We then use the results (4.1.29a) and (4.1.29b) in equation (4.1.27) to solve these integrals in the vacuum polarization tensor:

$$i\Pi_2^{\alpha\beta}(q) = -\frac{2ie^2}{(4\pi)^{d/2}} \int_0^1 \mathrm{d}x \left[-g^{\alpha\beta} \frac{\left(1 - \frac{d}{2}\right)\Gamma\left(1 - \frac{d}{2}\right)}{\Delta_m^{1-d/2}} + K^{\alpha\beta}(x,q) \frac{\Gamma\left(2 - \frac{d}{2}\right)}{\Delta_m^{2-d/2}} \right]. \tag{4.1.30}$$

Now we note that, by using the Γ-function property $\Gamma(1+z) = z\Gamma(z)$, the numerator of the first term in the square bracket can be written as $\Gamma(2 - d/2) = (1 - d/2)\Gamma(1 - d/2)$ so that the vacuum polarization tensor becomes

$$i\Pi_2^{\alpha\beta}(q) = -\frac{2ie^2}{(4\pi)^{d/2}} \lim_{\epsilon \to 0} \int_0^1 \mathrm{d}x \, \frac{\Gamma\left(2 - \frac{d}{2}\right)}{\Delta_m^{2-d/2}} \left[-g^{\alpha\beta}\Delta_m + K^{\alpha\beta}(x,q) \right], \tag{4.1.31}$$

without poles.

4.1.1.6 Final form of the vacuum polarization tensor

In order to get into the final form of the vacuum polarization tensor we recall the expression of $\Delta_m = m^2 - x(1-x)q^2$ given in (4.1.16) and of $K^{\alpha\beta} = g^{\alpha\beta}[m^2 + x(1-x)q^2] - 2x(1-x)q^\alpha q^\beta$ given in (4.1.18a). By replacing them in the equation (4.1.31) we arrive at the final form of the one-loop contribution to the vacuum polarization tensor:

$$i\Pi_2^{\alpha\beta}(q) = (g^{\alpha\beta}q^2 - q^\alpha q^\beta) \cdot i\Pi_2(q^2), \tag{4.1.32}$$

where

$$\Pi_2(q^2) = -\frac{4e^2}{(4\pi)^{d/2}} \int_0^1 \mathrm{d}x \, x(1-x) \frac{\Gamma\left(2 - \frac{d}{2}\right)}{m^2 - x(1-x)q^2}. \tag{4.1.33}$$

The equation (4.1.32) gives the tensor structure of the vacuum polarization tensor. We right see that if $i\Pi_2^{\alpha\beta}(q)$ is contract with q_α one gets the Ward identity: $q_\alpha \Pi_2^{\alpha\beta}(q) = 0$, meaning that the gauge invariance was preserved.

On the other hand, the equation (4.1.33) gives the one-loop correction to the complete photon propagator as (PESKIN; SCHROEDER, 1995, p. 246)

$$D^{\alpha\beta}(q^2) = -\frac{ig^{\alpha\beta}}{q^2[1-\Pi_2(q^2)]}. \tag{4.1.34}$$

From (4.1.34), we can see directly: If $\Pi_2(q^2)$ is regular at $q^2 = 0$ the propagator would have a pole at $q^2 = 0$, leaving then the photon non-massive; Otherwise, if $\Pi_2(q^2)$ contains a pole as $1/q^2$, the photon propagator now has a mass term and the photon acquires mass.

The QED (3+1) is notable in this regard. The quantity $\Pi_2(0)$ in this context has a finite value (despite a infinite term that must be removed within the renormalization framework), meaning that the photon remains non-massive after quantum corrections. Its expression is

$$\Pi_2(0) = -\frac{\alpha}{3\pi}\left[\lim_{\varepsilon\to 0}\frac{2}{\varepsilon} - \ln\left(\frac{m^2}{4\pi\lambda^2}\right) - \gamma\right], \tag{4.1.35}$$

where $\varepsilon = 4 - d$, λ is a mass parameter to preserver the dimensionless of coupling e [cf. footnote 3], α is the fine-structure constant, and γ is the Euler-Mascheroni constant.

4.1.1.7 Schwinger limit: $d \to 2$

The Schwinger model limit is occurs for $d \to 2$, in such a way that the equation (4.1.33) becomes

$$\Pi_2(q^2) = -\frac{e^2}{\pi}\int_0^1 \mathrm{d}x\, \frac{x(1-x)}{m^2 - x(1-x)q^2}. \tag{4.1.36}$$

To highlight the presence of a non-vanishing mass of photon we take one more limit: consider massless fermionic fields, i.e., $m \to 0$. Under this requirement the equation (4.1.36) reduces simply to

$$\Pi_2(q^2) = \frac{e^2}{\pi}\frac{1}{q^2}. \tag{4.1.37}$$

One can right see that there is a pole as $1/q^2$ making $\Pi_2(0)$ no longer regular.

So much so that when one uses (4.1.37) in (4.1.33) it can be seen that the photon propagator becomes

$$D^{\alpha\beta}(q^2) = -\frac{ig^{\alpha\beta}}{q^2 - \left(\frac{e^2}{\pi}\right)}. \tag{4.1.38}$$

Thus the pole of the propagator occurs for $q^2 = e^2/\pi$, which is just the dispersion relation of a massive particle, $p^2 = m_\gamma^2$. In other words, the (Schwinger) photon acquires a mass as

$$m_\gamma = \frac{e}{\sqrt{\pi}}. \tag{4.1.39}$$

4.1.2 Chiral Schwinger model

The Feynman diagram that originates the one-loop contribution to the vacuum polarization tensor in Chiral Schwinger model is the same as that shown in figure 4.1. What will change are the Feynman rules.

For the Lagrangian density given in equation (4.1.2) we have the following rule for interaction between the fermion field ψ and the gauge field A_α:

$$\text{vertex in Feynman diagrams} = -i\gamma^\alpha(g_v + g_a\gamma_5), \tag{4.1.40}$$

which is easily derived from the last term in the Lagrangian density in the same way as was done in section 3.2.1.

The other needed rule, as we know, is the free-fermionic propagator. This propagator can be obtained from the respective free Dirac equation

$$(i\gamma^\alpha\partial_\alpha - m - \mu\gamma_5)\psi(x) = 0. \tag{4.1.41}$$

The free-fermionic propagator[4] $S_5(x-y)$ is a formal solution for the equation (4.1.41) modified with a point source, i.e.,

$$(i\gamma^\alpha\partial_\alpha - m - \mu\gamma_5)S_5(x-y) = i\delta^4(x-y). \tag{4.1.42}$$

If one comes to the momentum space, the solution of the free-fermionic propagator has the following form

$$\widetilde{S}_5(p) = \lim_{\epsilon\to 0}\frac{i(\gamma^\alpha p_\alpha + m - \mu\gamma_5)}{p^2 - M^2 + i\epsilon}, \tag{4.1.43}$$

where

$$M = \sqrt{m^2 - \mu^2} \tag{4.1.44}$$

is the effective mass.

It is worth to note that the usual fermionic propagator given in equation (4.1.6) is recovered when one takes the Hermitian limit $\mu \to 0$ in the non-Hermitian fermionic propagator given in equation (4.1.43).

Thus each internal fermionic line in the diagram correponds to the propagator given in equation (4.1.43).

[4] The subscript "5" distinguishes the quantities of the Chiral Schwinger model from the usual Schwinger model.

The fermionic loop and momenta integration rules for the usual Schwinger model applied to this model.

By following these rules, one gets the amplitude corresponding to the one-loop contribution to the vacuum polarization tensor:

$$i\left(\Pi_2^{\alpha\beta}\right)_5(q) = -\lim_{\epsilon\to 0}\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{N_5^{\alpha\beta}(k,q)}{(k^2-M^2+i\epsilon)[(k+q)^2-M^2+i\epsilon]}, \tag{4.1.45}$$

where

$$N_5^{\alpha\beta}(k,q) = \mathrm{tr}\left\{\gamma^\alpha(g_v+g_a\gamma_5)(\gamma^\lambda k_\lambda + m - \mu\gamma_5)\gamma^\beta(g_v+g_a\gamma_5)[\gamma^\nu(k+q)_\nu + m - \mu\gamma_5]\right\}. \tag{4.1.46}$$

To find the correct result of this amplitude, we will apply dimensional regularization. However, due the presence of the matrix γ_5 the dimensional regularization gets in trouble because it is necessary to define this matrix in d dimensions and this is not an easy task (BAIKOV; IL'IN, 1991; NOVOTNÝ, 1994; CHANOWITZ; FURMAN; HINCHLIFFE, 1979; KÖRNER; KREIMER; SCHILCHER, 1992).

The canonical approach is to use the t'Hooft-Veltman prescription, for which the γ_5 is extended to d dimensions so that it anticommutes with γ^α for $\alpha = 0, 1$ and commutes with γ^α for $\alpha = 2, \ldots, d-1$.

However, we will not follow this procedure here. In fact, we will apply an easier prescription introduce by Thompson and Yu (1985). They present a general anticommutation relation between γ_5 and γ^α that works well in d dimensions, and in the limit $d \to 2$ it satisfies the well-known algebra of these matrices in this dimensionality.

This prescription allows to extend the vacuum polarization tensor given in equation (4.1.45) to d dimensions as

$$i\left(\Pi_2^{\alpha\beta}\right)_5(q) = -\lim_{\epsilon\to 0}\int \frac{\mathrm{d}^dk}{(2\pi)^d}\frac{N_5^{\alpha\beta}(k,q)}{(k^2-M^2+i\epsilon)[(k+q)^2-M^2+i\epsilon]}, \tag{4.1.47}$$

where

$$\begin{aligned} N_5^{\alpha\beta}(k,q) = \mathrm{tr}\big\{&(a\gamma^\alpha + b\theta^{\alpha\lambda}\gamma_\lambda\widetilde{\gamma}_5)(g_v+g_a\widetilde{\gamma}_5)(\gamma^\lambda k_\lambda + m - \mu\widetilde{\gamma}_5) \\ &\times (a\gamma^\beta + b\theta^{\beta\sigma}\gamma_\sigma\widetilde{\gamma}_5)(g_v+g_a\widetilde{\gamma}_5)[\gamma^\nu(k+q)_\nu + m - \mu\widetilde{\gamma}_5]\big\}. \end{aligned} \tag{4.1.48}$$

The $\theta^{\alpha\beta}$ is a second-rank antisymmetric tensor that belongs to the d-dimensional spacetime and reduces to the two-dimensional Levi-Civita tensor $\varepsilon^{\alpha\beta}$ in the limit $d \to 2$. The matrix $\widetilde{\gamma}_5$ is the n-dimensional version of the usual γ_5 which becomes γ_5 in the limit $d \to 2$. Finally, a and b are free-parameters.

Actually, what this prescription proposes is a change in the interaction vertex given by

$$\gamma^\alpha A_\alpha \to (a\gamma^\alpha + b\theta^{\alpha\beta}\gamma_\beta\widetilde{\gamma}_5)A_\alpha, \tag{4.1.49}$$

as can be seen in the numerator $N_5^{\alpha\beta}$ given in equation (4.1.48).

This proposal is feasible due to the fact that the V-A coupling may possibly break the gauge invariance of the quantum theory by anomalies. Therefore, there is no reason to insist in gauge invariance when one goes to d-dimensions (YU; YEUNG, 1987a,b).

Occurs that when one takes the limit $d \to 2$ and use the relation $\gamma_5\gamma^\alpha = \varepsilon^{\alpha\beta}\gamma_\beta$ appears a constraint upon the free-parameters given by

$$(a\gamma^\alpha + b\theta^{\alpha\beta}\gamma_\beta\widetilde{\gamma}_5)A_\alpha \underset{d\to 2}{\longrightarrow} (a-b)\gamma^\alpha A_\alpha \qquad \Longrightarrow \qquad a - b = 1. \tag{4.1.50}$$

It is worth to mention that this prescription was used to compute the vacuum polarization tensor of the Chiral Schwinger model with unitary couplings (YU; YEUNG, 1987a), to compute the axial anomaly for the usual QED (YU; YEUNG, 1987b), and to compute the vacuum polarization tensor of the axial model (BARCELOS-NETO; SOUZA, 1989).

Now we follow some steps to calculate the vacuum polarization tensor.

4.1.2.1 Rewriting the numerator

The numerator given in equation (4.1.48) can be written as

$$N_5^{\alpha\beta}(k,q) = N_1^{\alpha\beta\nu\rho}k_\nu(k_\rho + q_\rho) + N_2^{\alpha\beta}k_\nu + N_3^{\alpha\beta\rho}(k_\rho + q_\rho) + N_4^{\alpha\beta}, \tag{4.1.51}$$

where

$$N_1^{\alpha\beta\nu\rho} = \mathrm{tr}\left[(a\gamma^\alpha + b\theta^{\alpha\lambda}\gamma_\lambda\widetilde{\gamma}_5)(g_v + g_a\widetilde{\gamma}_5)\gamma^\nu(a\gamma^\beta + b\theta^{\beta\sigma}\gamma_\sigma\widetilde{\gamma}_5)(g_v + g_a\widetilde{\gamma}_5)\gamma^\rho\right], \tag{4.1.52a}$$

$$N_2^{\alpha\beta\nu} = \mathrm{tr}\left[(a\gamma^\alpha + b\theta^{\alpha\lambda}\gamma_\lambda\widetilde{\gamma}_5)(g_v + g_a\widetilde{\gamma}_5)\gamma^\nu(a\gamma^\beta + b\theta^{\beta\sigma}\gamma_\sigma\widetilde{\gamma}_5)(g_v + g_a\widetilde{\gamma}_5)(m - \mu\widetilde{\gamma}_5)\right], \tag{4.1.52b}$$

$$N_3^{\alpha\beta\rho} = \mathrm{tr}\left[(a\gamma^\alpha + b\theta^{\alpha\lambda}\gamma_\lambda\widetilde{\gamma}_5)(g_v + g_a\widetilde{\gamma}_5)(m - \mu\widetilde{\gamma}_5)(a\gamma^\beta + b\theta^{\beta\sigma}\gamma_\sigma\widetilde{\gamma}_5)(g_v + g_a\widetilde{\gamma}_5)\gamma^\rho\right], \tag{4.1.52c}$$

$$N_4^{\alpha\beta} = \mathrm{tr}\left[(a\gamma^\alpha + b\theta^{\alpha\lambda}\gamma_\lambda\widetilde{\gamma}_5)(g_v + g_a\widetilde{\gamma}_5)(m - \mu\widetilde{\gamma}_5)(a\gamma^\beta + b\theta^{\beta\sigma}\gamma_\sigma\widetilde{\gamma}_5)(g_v + g_a\widetilde{\gamma}_5)(m - \mu\widetilde{\gamma}_5)\right]. \tag{4.1.52d}$$

4.1.2.2 Feynman parametrization

We have seen in equation (4.1.13) that the Feynman parametrization for two parameters A and B in the denominator is given by

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \, \frac{1}{[Ax + (1-x)B]^2}. \tag{4.1.53}$$

If one sets $A = (k + q^2) - M^2 + i\epsilon$ and $B = k^2 - M^2 + i\epsilon$ then it is possible to get

$$\frac{1}{(k^2 - M^2 + i\epsilon)[(k+q)^2 - M^2 + i\epsilon]} = \int_0^1 \mathrm{d}x \, \frac{1}{(l^2 - \Delta_M + i\epsilon)^2}, \tag{4.1.54}$$

where

$$l = k + xq, \tag{4.1.55}$$

$$\Delta_M = M^2 - x(1-x)q^2. \tag{4.1.56}$$

Now we note that with the variable change (4.1.55) the numerator given in equation (4.1.51) becomes

$$\begin{aligned} N_5^{\alpha\beta}(l,x,q) = N_1^{\alpha\beta\nu\rho} l_\nu l_\rho &+ \Big[N_4^{\alpha\beta} - N_1^{\alpha\beta\nu\rho} x(1-x) q_\nu q_\rho - N_2^{\alpha\beta\nu} x q_\nu + N_3^{\alpha\beta\rho}(1-x) q_\rho\Big] \\ &+ \Big\{N_1^{\alpha\beta\nu\rho}\big[(1-x)q_\rho \delta_\nu^\tau - x q_\nu \delta_\rho^\tau\big] + N_2^{\alpha\beta\rho} + N_2^{\alpha\beta\rho}\Big\} l_\rho. \end{aligned} \tag{4.1.57}$$

By using the result (4.1.54) and (4.1.57) in the equation (4.1.47) the vacuum polarization tensor becomes

$$i\Big(\Pi_2^{\alpha\beta}\Big)_5(q) = -\lim_{\epsilon\to 0} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{N_5^{\alpha\beta}(l,x,q)}{(l^2 - \Delta_M + i\epsilon)^2}. \tag{4.1.58}$$

4.1.2.3 Reduction by symmetry

Now we can use the results found in section 4.1.1.3. One just make the replacement $m \to M$ in equations (4.1.21) and (4.1.24) to get

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^\lambda}{(l^2 - \Delta_M + i\epsilon)^2} = 0, \tag{4.1.59a}$$

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^\alpha l^\beta}{(l^2 - \Delta_M + i\epsilon)^2} = \frac{g^{\alpha\beta}}{d} \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^2}{(l^2 - \Delta_M + i\epsilon)^2}. \tag{4.1.59b}$$

Moreover, the equations (4.1.59a) and (4.1.59b) allows us to cast the equation (4.1.58) in the form

$$\begin{aligned} i\Big(\Pi_2^{\alpha\beta}\Big)_5(q) = -\lim_{\epsilon\to 0} \int_0^1 \mathrm{d}x \, \Big\{ &\Big[N_4^{\alpha\beta} - N_1^{\alpha\beta\nu\rho} x(1-x) q_\nu q_\rho - N_2^{\alpha\beta\nu} x q_\nu + N_3^{\alpha\beta\rho}(1-x) q_\rho\Big] I_M \\ &+ \frac{1}{d} g_{\nu\rho} N_1^{\alpha\beta\nu\rho} I_{l^2} \Big\}, \end{aligned} \tag{4.1.60}$$

where

$$I_M = \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{(l^2 - \Delta_M + i\epsilon)}, \tag{4.1.61a}$$

$$I_{l^2} = \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^2}{(l^2 - \Delta_M + i\epsilon)}. \tag{4.1.61b}$$

4.1.2.4 Wick rotation, solving integrals and final form of the vacuum polarization tensor

As it were made in sections 4.1.1.4 and 4.1.1.5, one can make the Wick rotation in the integrals given in equations (4.1.61a) and (4.1.61b) in such way that the denominator $l^2 - \Delta_M$ becomes $l_E^2 + \Delta_M$ so that we can perform the limit $\epsilon \to 0$, where l_E becomes a d-dimensional Euclidean vector.

After performing the Wick rotation we are left with two integrals suchlike that given in equations (4.1.29a) and (4.1.29b). Then we recover those results, add the corresponding imaginary factors due to the Wick rotation and make $M \to m$ to get

$$I_M = i \int \frac{\mathrm{d}^d l_E}{(2\pi)^d} \frac{1}{(l_E^2 + \Delta_M)^2} = \frac{i}{(4\pi)^{d/2}} \Gamma\left(2 - \frac{d}{2}\right) \frac{1}{\Delta_M^{2-d/2}}, \tag{4.1.62a}$$

$$I_{l^2} = -i \int \frac{\mathrm{d}^d l_E}{(2\pi)^d} \frac{l_E^2}{(l_E^2 + \Delta_M)^2} = -\frac{i}{(4\pi)^{d/2}} \left(\frac{d}{2}\right) \Gamma\left(1 - \frac{d}{2}\right) \frac{1}{\Delta_M^{1-d/2}}. \tag{4.1.62b}$$

Hence, using the results (4.1.62a) and (4.1.62b) in the equation (4.1.60) we obtain the final form of the vacuum polarization tensor:

$$i\left(\Pi_2^{\alpha\beta}\right)_5(q) = -i \int_0^1 \mathrm{d}x \, (A^{\alpha\beta} + B^{\alpha\beta}), \tag{4.1.63}$$

where

$$A^{\alpha\beta} = -\frac{1}{2} g_{\nu\rho} N_1^{\alpha\beta\nu\rho} \frac{1}{(4\pi)^{d/2}} \frac{\Gamma\left(1 - \frac{d}{2}\right)}{\Delta_M^{1-d/2}}, \tag{4.1.64a}$$

$$B^{\alpha\beta} = \left[N_4^{\alpha\beta} - N_1^{\alpha\beta\nu\rho} x(1-x) q_\nu q_\rho - N_2^{\alpha\beta\nu} x q_\nu + N_3^{\alpha\beta\rho}(1-x) q_\rho\right] \frac{1}{(4\pi)^{d/2}} \frac{\Gamma\left(2 - \frac{d}{2}\right)}{\Delta_M^{2-d/2}}. \tag{4.1.64b}$$

4.1.2.5 Schwinger limit: $d \to 2$

We firstly observe that the term $B^{\alpha\beta}$ is regular under the limit $d \to 2$ so that it can be calculated directly in two-dimensional spacetime.

However, to make this calculation it is necessary to find out the expressions for $N_1^{\alpha\beta\nu\rho}$, $N_2^{\alpha\beta\rho}$, $N_3^{\alpha\beta\rho}$ and $N_4^{\alpha\beta}$ when the limit $d \to 2$ is taken. It can be shown that these quantities given respectively in equations (4.1.52a) to (4.1.52d) produce the following results

$$N_1^{\alpha\beta\nu\rho} \underset{d\to 2}{\longrightarrow} 2(g_v^2+g_a^2)(g^{\alpha\nu}g^{\beta\rho}-g^{\alpha\beta}g^{\nu\rho}+g^{\alpha\rho}g^{\nu\beta})-4g_vg_a(\epsilon^{\alpha\nu}g^{\beta\rho}-\epsilon^{\alpha\beta}g^{\nu\rho}+\epsilon^{\alpha\rho}g^{\nu\beta}), \tag{4.1.65a}$$

$$N_2^{\alpha\beta\nu} \underset{d\to 2}{\longrightarrow} 0, \tag{4.1.65b}$$

$$N_3^{\alpha\beta\nu} \underset{d\to 2}{\longrightarrow} 0, \tag{4.1.65c}$$

$$N_4^{\alpha\beta} \underset{d\to 2}{\longrightarrow} (g_v^2-g_a^2)M^2g^{\alpha\beta}. \tag{4.1.65d}$$

By taking then the limit $d \to 2$ in equation (4.1.64b) and by using the results given in equations (4.1.65a) to (4.1.65d) we find thus

$$B^{\alpha\beta} \underset{d\to 2}{\longrightarrow} \frac{1}{4\pi}\Big(\big\{2\big[(g_v^2+g_a^2)g^{\alpha\beta}-2g_vg_a\epsilon^{\alpha\beta}\big]q^2-4\big[(g_v^2+g_a^2)g^{\alpha\nu}-2g_vg_a\epsilon^{\alpha\nu}\big]q_\nu q^\beta\big\}x(1-x) \\ +(g_v^2-g_a^2)M^2g^{\alpha\beta}\Big)\frac{1}{\Delta_M}, \tag{4.1.66}$$

where the trace properties of γ-matrices respects exactly those of the two-dimensional spacetime.

On the other hand, the term $A^{\alpha\beta}$ has a pole $\Gamma(0)$ when the limit $d \to 2$ is taken. So we need to calculate the quantity $N_1^{\alpha\beta\nu\rho}$ in d dimensions so that the trace properties of γ-matrices cancels out this pole.

If one recalls the $N_1^{\alpha\beta\nu\rho}$ given in equation (4.1.52a) and use the anticommutivity of $\widetilde{\gamma}_5$ with the other γ-matrices in $A^{\alpha\beta}$ it is possible to obtain

$$A^{\alpha\beta} = -\frac{\Gamma\big(1-\frac{d}{2}\big)}{2(4\pi)^{\frac{d}{2}}\Delta_M^{1-\frac{d}{2}}}\,\mathrm{tr}\big[(a\gamma^\alpha+b\theta^{\alpha\lambda}\gamma_\lambda\widetilde{\gamma}_5)(g_v+g_a\widetilde{\gamma}_5)(a\gamma_\rho\gamma^\beta\gamma^\rho-b\theta^{\beta\sigma}\gamma_\rho\gamma_\sigma\gamma^\rho\widetilde{\gamma}_5)(g_v-g_a\widetilde{\gamma}_5)\big]. \tag{4.1.67}$$

However, the γ-matrices satisfy the following contraction property in d dimensions: $\gamma_\rho\gamma_\sigma\gamma^\rho = (2-d)\gamma_\sigma$. Then we can remove the pole by using a Γ-function property, namely $\Gamma(1+z) = z\Gamma(z)$, and get

$$A^{\alpha\beta} = -\frac{\Gamma\big(2-\frac{d}{2}\big)}{(4\pi)^{\frac{d}{2}}\Delta_M^{1-\frac{d}{2}}}\,\mathrm{tr}\big[(a\gamma^\alpha+b\theta^{\alpha\lambda}\gamma_\lambda\widetilde{\gamma}_5)(g_v+g_a\widetilde{\gamma}_5)(a\gamma^\beta-b\theta^{\beta\sigma}\gamma_\sigma\widetilde{\gamma}_5)(g_v-g_a\widetilde{\gamma}_5)\big]. \tag{4.1.68}$$

Since the pole was removed we can now evaluate $A^{\alpha\beta}$ correctly in the two-dimensional spacetime. Hence, performing this limit, using the γ-matrices trace properties, and the condition $(a-b) = 1$, one gets

$$A^{\alpha\beta} \underset{d\to 2}{\longrightarrow} -\frac{2}{4\pi}(2a-1)\big[(g_v^2+g_a^2)g^{\alpha\beta} - 2g_v g_a \epsilon^{\alpha\beta}\big]. \tag{4.1.69}$$

Now it just a matter of gather the relevant contributions: Take the results in equations (4.1.66) and (4.1.69) and use them in (4.1.63), which after some straightforward manipulation it can be written as

$$\begin{aligned} i\Big(\Pi_2^{\alpha\beta}\Big)_5(q) = &-\frac{i}{4\pi}\int_0^1 \mathrm{d}x\,\frac{1}{M^2 - x(1-x)q^2}\Big\{2\big[(g_v^2+g_a^2)g^{\alpha\beta} - 2g_v g_a\epsilon^{\alpha\beta}\big] \\ &\times\left[2ax(1-x)q^2 - (2a-1)M^2\right] - 4x(1-x)\big[(g_v^2+g_a^2)g^{\alpha\nu} - 2g_v g_a \epsilon^{\alpha\nu}\big]q_\nu q^\beta \\ &+ (g_v^2 - g_a^2)M^2 g^{\alpha\beta}\Big\}, \end{aligned} \tag{4.1.70}$$

where it was used $\Delta_M = M^2 - x(1-x)q^2$.

Finally we consider the limit for which the fermion becomes non-massive, i.e., $M \to 0$. Under this restriction, the equation (4.1.70) reduces to

$$i\Big(\Pi_2^{\alpha\beta}\Big)_5(q) = \frac{i}{\pi q^2}g_v^2(1+r^2)\left(g^{\alpha\nu} - \frac{2r}{1+r^2}\epsilon^{\alpha\nu}\right)\big(a\delta_\nu^\beta q^2 - q_\nu q^\beta\big), \tag{4.1.71}$$

where

$$r = \frac{g_a}{g_v}. \tag{4.1.72}$$

It is worth noting that the unitary coupling limits applied into the equation (4.1.71), for which $g_v, g_a \to 1$, respect the result found by Yu and Yeung (1987a).

Now, besides the tensor structure in equation (4.1.71) we observe the quantity

$$(\Pi_2)_5(q^2) = \frac{g_v^2(1+r^2)}{\pi}\frac{1}{q^2} = \frac{(g_v^2+g_a^2)}{\pi}\frac{1}{q^2}. \tag{4.1.73}$$

We thus see that, analogously to the usual Schwinger model, this quantity has a pole in q^2. This would lead us to think that the gauge field acquires a mass given by

$$(m_\gamma)_5 = \sqrt{\frac{(g_v^2+g_a^2)}{\pi}}. \tag{4.1.74}$$

However, for a better comprehension of the phenomenon of mass generation in this model would be necessary to perform a bosonization (JACKIW; RAJARAMAN, 1985; RYANG, 1987).

Finally, we observe that when one takes $g_v \to e$ and $g_a \to 0$, the equations (4.1.71), (4.1.73) and (4.1.74) reduce respectively to the equations (4.1.32), (4.1.37) and (4.1.39), that is, the results of the Chiral Schwinger model reduce to those of the Schwinger model.

4.2 Chiral anomaly

It is well-known that some classical symmetries are broken when the quantization procedure is done. One fabulous example is the chiral or axial symmetry in the usual QED: One performs a chiral transformation of the fermion fields, calculate the continuity equation for the axial-vector current, and realizes that the quantization gives an extra factor to that classical continuity equation, which is called chiral anomaly (FUJIKAWA; SUZUKI, 2004, p. 62).

So it is expected the anomalies to have a great importance in theories and models, given that they break the classical continuity equation for the axial-vector current, also known as Ward identity.

In fact, its importance is twofold: (i) Anomalies are good for experiment in case if the underlying symmetry is an external symmetry, because then they are responsible for. e.g., the particle properties, the physics of the particle decay or transitions; and (ii) Anomalies are bad for theory since the anomalous contribution to Ward identity spoils the renormalizability of the gauge theory or even the unitarity of the scattering matrix, and this happens if the underlying symmetry is an internal symmetry (BERTLMANN, 1996, p. 244–245).

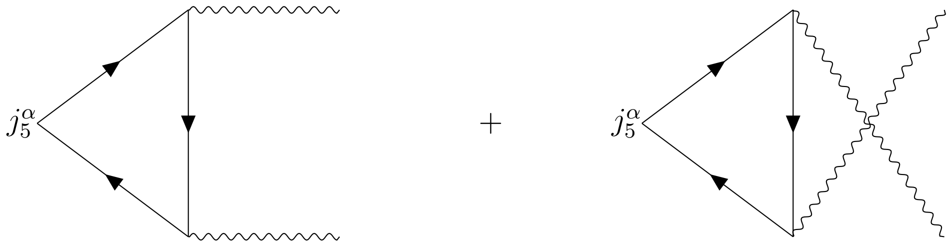
Some examples of the experimental importance of the anomalies are: (i) The decay $\pi^0 \to \gamma\gamma$, which it is determined by the Adler-Bell-Jackiw anomaly, (ii) The spin of the proton departs from its expectation value due to the anomaly, and (iii) The reaction $\gamma \to \pi^+\pi^-\pi^0$ receives contribution from the non-Abelian anomaly.

As theoretical implications from anomalies we can mention the following: (i) In vector-like models where all fermions couple symmetrically in the left and right sectors, the left gauge anomalies cancel out the right ones, (ii) The anomaly in the Standard Model $SU(2) \times U(1)$ cancels due to the arrangement of the fermions in left-handed doublets and right-handed singlets, so that the absence of anomalies restricts the fermionic content of the theory quite severely, and (iii) Nonlocal counterterm of gauge fields introduced in the action can cancels the anomaly (BERTLMANN, 1996, p. 245–247).

Having seen some examples highlighting the importance of anomalies in models and experiments, it is worthwhile to calculate the chiral anomaly in our non-Hermitian QED with $\mathcal{PT}$ symmetry, and this will be done for the Chiral Schwinger model with non-unitary coupling presented in section 4.2.3.2.

There are two approaches to calculate the chiral anomaly[5]: perturbatively and non-perturbatively. The first one makes use of the Feynman diagrams where it is needed to calculate the well-know triangle diagrams shown in figure 4.2 (FUJIKAWA; SUZUKI, 2004, p. 58 and 59).

Figure 4.2 – The two one-loop Feynman diagrams contributing to the chiral anomaly in QED.



Source: Authors (2024).

The other approach makes use of the path integral description, for which the path integral measure is related to the chiral anomaly. In fact, the chiral anomaly revels itself in the Jacobian of the path integral measure under the chiral transformation (FUJIKAWA; SUZUKI, 2004, p. 71).

We will follow the non-perturbative approach for two reasons: (i) Due to its elegance, and (ii) Perturbatively calculations are already made in the section 3.2.1, so it is a good idea to give it a variety.

Our starting point is the Euclidean path integral of the QED-like model in an arbitrary d dimensions,[6] which is just the exponential of the action of the QED measured over the fermion and gauge fields[7]:

$$Z[\psi, \bar{\psi}, A_\alpha] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi[\mathcal{D}A_\alpha]e^{S_{\text{QED-like}}}, \tag{4.2.1}$$

where

$$S_{\text{QED-like}} = \int \mathrm{d}^d x \left[-\frac{1}{4} F_{\alpha\beta}F^{\alpha\beta} + \bar{\psi}(i\not{D} - m - \mu\gamma_5)\psi \right], \tag{4.2.2}$$

where $\not{D}$ stands for the covariant derivative or Dirac operator of the model, which is usually Hermitian, and μ stands for a possibility of including a axial deformation in the model.

[5] There is another way, which is calculate the divergence of the axial-vector current directly via covariant regularization, cf. Fujikawa and Suzuki (2004, sec. 4.3).

[6] As long as it is possible to define the axial matrix γ_5.

[7] It is worthwhile to mention that in the path integral measure of the gauge field $[\mathcal{D}A_\alpha]$ is included a suitable gauge fixing term.

4.2.1 Classical current conservation

Firstly, we will obtain from the QED-like path integral the classical continuity equation for the axial-vector current.

To achieve this objective, we initiate with a infinitesimal local chiral transformation of the fermionic field:

$$\psi'(x) = e^{i\alpha(x)\gamma_5}\psi(x) \approxeq \psi(x) + i\alpha(x)\gamma_5\psi(x), \tag{4.2.3a}$$

$$\bar{\psi}'(x) = \bar{\psi}(x)e^{i\alpha(x)\gamma_5} \approxeq \bar{\psi}(x) + i\alpha(x)\bar{\psi}(x)\gamma_5, \tag{4.2.3b}$$

where $\alpha(x)$ is a scalar function depending on the spacetime coordinates.

Thus we must have

$$Z[\psi', \bar{\psi}', A_\alpha] = Z[\psi, \bar{\psi}, A_\alpha], \tag{4.2.4}$$

giving that the value of a definite integral does not depends on the naming of integration variables (FUJIKAWA; SUZUKI, 2004, p. 49).

However, the chiral transformation imposes that the fermionic measure changes according to a Jacobian $\mathcal{J}$:

$$\mathcal{D}\bar{\psi}'\mathcal{D}\psi' = \mathcal{J}\mathcal{D}\bar{\psi}\mathcal{D}\psi. \tag{4.2.5}$$

Let us then assume that the Jacobian of the fermioninc measure under this chiral transformation is the unity, that is,

$$\mathcal{J} = 1. \tag{4.2.6}$$

Under by the assumption (4.2.6), and by using the equations (4.2.1) to (4.2.3b) in the equation (4.2.4), it is possible to find the relation that establishes the classical continuity equation for the the axial-vector current[8]:

$$\partial_\alpha \left\langle \bar{\psi}\gamma^\alpha\gamma_5\psi \right\rangle = 2im \left\langle \bar{\psi}\gamma_5\psi \right\rangle + 2i\mu \left\langle \bar{\psi}\psi \right\rangle, \tag{4.2.7}$$

where was used the notation

$$\langle O \rangle = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi[\mathcal{D}A_\alpha]Oe^{S_{\text{QED-like}}}, \tag{4.2.8}$$

in which O is an arbitrary operator.

[8] The partial derivative ∂_α comes from the action of the covariant derivative D_α on the exponential in the chiral transformation, given that, generally, $D_\alpha = \partial_\alpha + \Gamma A_\alpha$, where Γ is a spinorial structure.

Now is the time to make some remarks. Firstly, we observe that the dimension of the spacetime is irrelevant to the calculus. Thus the result in the equation (4.2.7) stands for any dimension d in which the matrix γ_5 is well-defined.

So we can establish the classical continuity equation for the axial-vector current in the usual QED (1+1) and the non-Hermitian QED (1+1). For the first, we just takes the limit in which the axial deformation goes to zero, that is, $\mu \to 0$, so that the equation (4.2.7) becomes

$$\partial_\alpha \left\langle \bar{\psi}\gamma^\alpha\gamma_5\psi \right\rangle = 2im \left\langle \bar{\psi}\gamma_5\psi \right\rangle . \tag{4.2.9}$$

It is worth noting that in the non-massive fermionic limit, for which $m \to 0$, the QED (1+1) becomes the Schwinger model and the right-hand side of (4.2.9) vanishes so that the axial-vector current is conserved, allowing us to say that the Schwinger model is chiral invariant (FUJIKAWA; SUZUKI, 2004, p. 58).

For the latter, we just keep the axial deformation, so that the equation (4.2.7) remains the same:

$$\partial_\alpha \left\langle \bar{\psi}\gamma^\alpha\gamma_5\psi \right\rangle = 2im \left\langle \bar{\psi}\gamma_5\psi \right\rangle + 2i\mu \left\langle \bar{\psi}\psi \right\rangle . \tag{4.2.10}$$

The non-massive fermionic limit in the model with axial deformation is achieved by setting $\mu = \pm m$, which gives

$$\partial_\alpha \left\langle \bar{\psi}\gamma^\alpha\gamma_5\psi \right\rangle = 2im \big[\left\langle \bar{\psi}\gamma_5\psi \right\rangle \pm \left\langle \bar{\psi}\psi \right\rangle \big], \tag{4.2.11}$$

but this does not make the right-hand side of (4.2.11) vanishes in general, only when $\left\langle \bar{\psi}\gamma_5\psi \right\rangle = \mp \left\langle \bar{\psi}\psi \right\rangle$. Thus the axial-vector current is not generally conserved and the Chiral Schwinger model is not chiral invariant.

4.2.2 Breaking of classical continuity equation

Now we will show how classical continuity equation is modified due the chiral anomaly, which originates from the assumption that the Jacobian of the fermionic measure under chiral transformation is not the unity.

Firstly, we expand the Dirac field into eigenfunctions of the hermitian operator $\not{D}$:

$$\psi'(x) = \sum_n a_n \varphi_n(x), \tag{4.2.12a}$$

$$\bar{\psi}'(x) = \sum_m \bar{b}_m \varphi_m^\dagger(x), \tag{4.2.12b}$$

where the coefficients a_n and $\bar{b}_m$ are independent Grassmann variables.

The Dirac operator satisfies

$$\not{D}\varphi_n(x) = \lambda_n \varphi_n(x), \tag{4.2.13}$$

where λ_n are eigenvalues of $\not{D}$ and the set of eigenfunctions are orthonormal and complete:

$$\int \mathrm{d}^d x \, \varphi_m^\dagger(x)\varphi_n(x) = \delta_{nm}, \tag{4.2.14a}$$

$$\sum_n \varphi(x)\varphi_n^\dagger(y) = \delta^d(x-y). \tag{4.2.14b}$$

We observe that this basis set allow us to diagonalize the Dirac action:

$$\int \mathrm{d}^d x \, \bar{\psi}(i\not{D} - m)\psi = \sum_n (i\lambda_n - m)\bar{b}_n a_n. \tag{4.2.15}$$

Now we look at the equations (4.2.12a) and (4.2.12b) and see that this expansion implies in the following measure transformation:

$$\mathcal{D}\psi = [\det \varphi_n]^{-1} \prod_n \mathrm{d}a_n \,, \tag{4.2.16a}$$

$$\mathcal{D}\bar{\psi} = [\det \varphi_m^\dagger]^{-1} \prod_m \mathrm{d}\bar{b}_m \,, \tag{4.2.16b}$$

where the (-1) exponent comes from the rule of measure transformation for Grassmann variables (BERTLMANN, 1996, p. 256).

We then see that the equations (4.2.16a) and (4.2.16b) implies

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi = \prod_n \mathrm{d}\bar{b}_n \, \mathrm{d}a_n \,, \tag{4.2.17}$$

in wish was used the orthonormalization relation.

The equation (4.2.17) tells us that the fermionic path measure can be converted in the measure over all the coefficients from the Dirac field expansion (4.2.12a) and (4.2.12b).

Now one uses the expansion equation (4.2.12a) in the infinitesimal local chiral transformation equation (4.2.3a), multiplies it by $\varphi_m^\dagger(x)$ on the left and integrate over the spacetime coordinates. We also do the same thing for the equation (4.2.3b) but now using equation (4.2.12b), multiplying it by $\varphi_n(x)$ and integrate over the spacetime coordinates. Then we get

$$a_n' = \sum_m C_{nm} a_m, \quad \bar{b}_n' = \sum_m \bar{b}_m C_{nm}, \tag{4.2.18}$$

where

$$C_{nm} = \delta_{nm} + i \int \mathrm{d}^d x \, \varphi_n^\dagger(x)\alpha(x)\gamma_5\varphi_m(x). \tag{4.2.19}$$

Now it is easy to see that the fermionic path integral measure is given by

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi = \mathcal{J}[\alpha(x)] \prod_n \mathrm{d}\bar{b}_n \, \mathrm{d}a_n \,, \tag{4.2.20}$$

where $\mathcal{J}$ is the Jacobian given by

$$\mathcal{J}[\alpha(x)] = (\det C)^{-2}. \tag{4.2.21}$$

However, it is well-known that the determinant of an operator A can be written as $\det A = e^{\mathrm{tr}\ln A}$. When this property is used in equation (4.2.21), and taking into account the fact that we are considering infinitesimal chiral transformation, for which $\ln(1 + f[\alpha(x)]) = f[\alpha(x)] + \mathcal{O}(\alpha^2(x))$, the following result for the Jacobian is reached:

$$\ln \mathcal{J}[\alpha(x)] = -\int \mathrm{d}^d x \, \alpha(x)\mathcal{A}, \tag{4.2.22}$$

where $\mathcal{A}$ is identified as the chiral anomaly given by

$$\mathcal{A} = 2i \sum_n \varphi_n^\dagger(x)\gamma_5\varphi_n(x) \tag{4.2.23}$$

Actually, the sum given in equation (4.2.23) is not regular. Indeed, if one recognizes that an arbitrary operator O obeys the relation $\sum_n \varphi_n^\dagger(x) O \varphi_n(y) = \mathrm{tr}(O)\delta^d(x-y)$, there is no difficult in shown that

$$\mathcal{A} = 2i \sum_n \varphi_n^\dagger(x)\gamma_5\varphi_n(x) = 2i\,\mathrm{tr}(\gamma_5)\delta^d(0) \to 0 \cdot \infty, \tag{4.2.24}$$

where it was used the completeness relation of the eigenfunction give in equation (4.2.14b).

So it is necessary to regularize this sum to get the correct chiral anomaly. Here we follow Fujikawa's approach (FUJIKAWA; SUZUKI, 2004, p. 68–69), where it is considered an arbitrary function that is smooth and decreasing sufficiently rapidly at infinity

$$f\left(\frac{\lambda_n^2}{\Lambda^2}\right), \tag{4.2.25}$$

with $\Lambda \to \infty$, and that obeys

$$f(\infty) = f'(\infty) = f''(\infty) = \cdots = 0, \quad f(0) = 1. \tag{4.2.26}$$

Now it is just a matter of expand the eigenfunctions $\varphi_n(x)$ in their Fourier representation (plane waves) to get the regularized expression for the chiral anomaly:

$$\mathcal{A} = 2i \sum_n \varphi_n^\dagger(x)\gamma_5\varphi_n(x) = 2i \lim_{\Lambda\to\infty} \mathrm{tr} \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik\cdot x} f\left(\frac{\not{D}^2}{\Lambda^2}\right) e^{ik\cdot x}, \tag{4.2.27}$$

where it was used the completeness relation of the eigenfunction give in equation (4.2.14b).

To proceed with this development it is necessary to define the spacetime dimension we are dealing on and to introduce the expression of the Dirac operator $\not{D}$, which depends on the model with we working on.

For (1+1) spacetimes models, it is sufficient to expand the function f to the order Λ^{-2} since after performing a suitable variable change there is no contributions due to higher orders. For (3+1) spacetime models, we expand the function f to the order Λ^{-4} for the same reason as before.

Before proceeding to the calculations of the chiral anomaly for the usual Schwinger model and the Chiral Schwinger model, it can be shown that the Jacobian given in (4.2.22) modifies the continuity equation (4.2.7) as

$$\partial_\alpha \left\langle \bar{\psi}\gamma^\alpha\gamma_5\psi \right\rangle = 2im \left\langle \bar{\psi}\gamma_5\psi \right\rangle + 2i\mu \left\langle \bar{\psi}\psi \right\rangle + \mathcal{A}. \tag{4.2.28}$$

Thus the chiral anomaly enters directly on the continuity equation. We then say that the chiral anomaly gives a quantum correction to the classical continuity equation, since it appears after the quantization procedure of the fermionic field in terms of the eigenfunctions of the Dirac operator. We will refer to this modified continuity equation as quantum continuity equation.

4.2.3 (1+1) spacetime

Likewise the section 3.2, we introduce here the anomaly firstly for the usual Schwinger model and the replicate the result for the Schwinger chiral model with non-unitary couplings.

4.2.3.1 Schwinger model

For the usual Schwinger Model, the Dirac operator is given by

$$\not{D} = \gamma^\alpha D_\alpha = \gamma^\alpha(\partial_\alpha - ieA_\alpha), \tag{4.2.29}$$

which can be shown that it is Hermitian, given that in the Euclidean spacetime the γ-matrices are anti-Hermitian: $(\gamma^\alpha)^\dagger = -\gamma^\alpha$; and the metric is just $g^{\alpha\beta} = -\delta^{\alpha\beta}$.

Now we note that $\not{D}^2 = \gamma^\alpha\gamma^\beta D_\alpha D_\beta$. Then we use the γ-matrices relation $\gamma^\alpha\gamma^\beta = g^{\alpha\beta} + (1/2)[\gamma^\alpha, \gamma^\beta]$ and the commutation relation $[D_\alpha, D_\beta] = -ieF_{\alpha\beta}$ to write

$$\not{D}^2 = D_\alpha D^\alpha - \frac{ie}{4}[\gamma^\alpha, \gamma^\beta]\, F_{\alpha\beta}. \tag{4.2.30}$$

We then return to the equation (4.2.27), set $d = 2$, use equation (4.2.30) in it, apply the argument of function f to the positive exponential and perform the change $k_\alpha \to \Lambda k_\alpha$ to get

$$\mathcal{A} = 2i \lim_{\Lambda\to\infty} \Lambda^2 \operatorname{tr} \int \frac{\mathrm{d}^2k}{(2\pi)^2}\gamma_5 f\left(-k_\alpha k^\alpha + \frac{2i}{\Lambda}k_\alpha D^\alpha + \frac{1}{\Lambda^2}D_\alpha D^\alpha - \frac{ie}{4\Lambda^2}[\gamma^\alpha,\gamma^\beta]\, F_{\alpha\beta}\right). \tag{4.2.31}$$

Moreover, we consider the Taylor expansion of the function f around the value $x = -k_\alpha k^\alpha = k^2$ and stop at the order $\mathcal{O}(\Lambda^{-2})$, since higher order terms will vanish in the limit $\Lambda \to \infty$. Hence, by taking all these considerations into account we find

$$\mathcal{A} = 2i \lim_{\Lambda\to\infty} \Lambda^2 \int \frac{\mathrm{d}^2k}{(2\pi)^2}\big[f_0 + f_1 + f_2 + \mathcal{O}\big(\Lambda^{-3}\big)\big], \tag{4.2.32}$$

where

$$f_0 = f(k^2)(\operatorname{tr}\gamma_5), \tag{4.2.33a}$$

$$f_1 = f'(k^2)\left[\frac{2i}{\Lambda}(\operatorname{tr}\gamma_5)k_\alpha D^\alpha + \frac{1}{\Lambda^2}(\operatorname{tr}\gamma_5)D_\alpha D^\alpha - \frac{ie}{4\Lambda^2}(\operatorname{tr}\gamma_5[\gamma^\alpha,\gamma^\beta]\,)F_{\alpha\beta}\right], \tag{4.2.33b}$$

$$f_2 = \frac{1}{2}f''(k^2)\left[-\frac{4}{\Lambda^2}(\operatorname{tr}\gamma_5)k_\alpha k_\beta D^\alpha D^\beta\right] + \mathcal{O}\big(\Lambda^{-3}\big). \tag{4.2.33c}$$

We then use the trace properties (FUJIKAWA; SUZUKI, 2004, p. 153)

$$\operatorname{tr}\gamma_5 = 0, \quad \operatorname{tr}\gamma_5[\gamma^\alpha,\gamma^\beta] = 4i\epsilon^{\alpha\beta}, \tag{4.2.34}$$

where $\epsilon^{\alpha\beta}$ is the two-dimensional Levi-Civita tensor.

In terms of the property (4.2.34), the equations (4.2.33a) to (4.2.33c) are reduced to

$$f_0 = 0, \quad f_1 = \frac{e}{\Lambda^2}\epsilon^{\alpha\beta}F_{\alpha\beta}f'(k^2), \quad f_2 = \mathcal{O}\big(\Lambda^{-3}\big), \tag{4.2.35}$$

which are used in the chiral anomaly (4.2.32) to give

$$\mathcal{A} = 2ie\epsilon^{\alpha\beta}F_{\alpha\beta}\int \frac{\mathrm{d}^2k}{(2\pi)^2}f'(k^2), \tag{4.2.36}$$

where the limit $\Lambda \to \infty$ was taken.

Finally we solve the integral present in the chiral anomaly $\mathcal{A}$ by using the properties of the function f given in equation (4.2.26):

$$\int \frac{\mathrm{d}^2k}{(2\pi)^2}f'(k^2) = \frac{1}{(2\pi)^2}\int \mathrm{d}\Omega_2 \int_0^\infty \mathrm{d}k\,|k| f'(k^2) = \frac{1}{4\pi}\int_0^\infty \mathrm{d}k^2\, f'(k^2) = \frac{1}{4\pi}\, f(k^2)\bigg|_0^\infty = -\frac{1}{4\pi}. \tag{4.2.37}$$

At last, replacing the result (4.2.37) in the equation (4.2.36) we find the corresponding chiral anomaly for the Schwinger model:

$$\mathcal{A} = -\frac{ie}{2\pi}\epsilon^{\alpha\beta}F_{\alpha\beta}. \tag{4.2.38}$$

Moreover, we use the (4.2.38) in the (4.2.28) to get the quantum continuity equation for the QED (1+1):

$$\partial_\alpha(\bar{\psi}\gamma^\alpha\gamma_5\psi) = 2im(\bar{\psi}\gamma_5\psi) - 2i\Big(\frac{e}{4\pi}\epsilon^{\alpha\beta}F_{\alpha\beta}\Big), \tag{4.2.39}$$

or for the Schwinger model, in which $m \to 0$,

$$\partial_\alpha(\bar{\psi}\gamma^\alpha\gamma_5\psi) = -2i\Big(\frac{e}{4\pi}\epsilon^{\alpha\beta}F_{\alpha\beta}\Big). \tag{4.2.40}$$

It can then be seen that the Schwinger model is not longer chiral invariant after the quantization procedure, given that the right hand side of (4.2.40) is non null.

4.2.3.2 Chiral Schwinger model with non-unitary couplings

For the Chiral Schwinger model, the Dirac operator can be read from (4.1.2) as

$$\nabla\!\!\!/ = \gamma^\alpha\nabla_\alpha = \gamma^\alpha[\partial_\alpha - i(g_v + g_a\gamma_5)A_\alpha] = \gamma^\alpha D_\alpha - ig_a\gamma^\alpha\gamma_5 A_\alpha, \tag{4.2.41}$$

where D_α is given by (4.2.29).

Moreover, one can shown that the operator $\nabla\!\!\!/$ is Hermitian, given that in the Euclidean spacetime the γ-matrices are anti-Hermitian, $(\gamma^\alpha)^\dagger = -\gamma^\alpha$, and the axial matrix γ_5 is Hermitian: $\gamma_5^\dagger = \gamma_5$.

Now we observe that $\nabla\!\!\!/^2 = \gamma^\alpha\gamma^\beta\nabla_\alpha\nabla_\beta$. We can rewrite this expression if terms of the γ-matrices relation $\gamma^{\alpha\beta} = g^{\alpha\beta} + (1/2)[\gamma^\alpha, \gamma^\beta]$, the anti-commutation relation $[\gamma_5, \gamma_\alpha] = 0$, and the following relations: (i) $[\gamma^\alpha, \gamma^\beta]\, D_\alpha D_\beta = -(i/2)g_v[\gamma^\alpha, \gamma^\beta]\, F_{\alpha\beta}$, (ii) $[\gamma^\alpha, \gamma^\beta]\, A_\alpha A_\beta = 0$, (iii) $A_\alpha D_\beta[\gamma^\alpha, \gamma^\beta] = A_\alpha\partial_\beta[\gamma^\alpha, \gamma^\beta]$, and (iv) $(D_\alpha A_\beta)[\gamma^\alpha, \gamma^\beta] = (1/2)F_{\alpha\beta}[\gamma^\alpha, \gamma^\beta]$. The result is

$$\nabla\!\!\!/^2 = D_\alpha D^\alpha - \frac{i}{4}[\gamma^\alpha, \gamma^\beta]\, F_{\alpha\beta}(g_v + g_a\gamma_5) - ig_a(D_\alpha A^\alpha)\gamma_5 + ig_a A_\alpha\partial_\beta[\gamma^\alpha, \gamma^\beta]\, \gamma_5 + g_a^2 A_\alpha A^\alpha, \tag{4.2.42}$$

where the parenthesis means that the derivative application is only on the terms inside it.

Now we can set the Lorentz gauge condition, for which $\partial_\alpha A^\alpha = 0$, which implies $(D_\alpha A^\alpha) = -ig_v A_\alpha A^\alpha$, in such a way that the equation (4.2.42) becomes

$$\nabla\!\!\!/^2 = D_\alpha D^\alpha - \frac{i}{4}[\gamma^\alpha, \gamma^\beta]\, F_{\alpha\beta}(g_v + g_a\gamma_5) + ig_a A_\alpha\partial_\beta[\gamma^\alpha, \gamma^\beta]\, \gamma_5 + g_a(g_a - g_v\gamma_5)A_\alpha A^\alpha. \tag{4.2.43}$$

However, in the (1+1) spacetime the γ-matrices and the axial matrix γ_5 are related in terms of (FUJIKAWA; SUZUKI, 2004, p. 153)

$$[\gamma^\alpha, \gamma^\beta] = 2i\epsilon^{\alpha\beta}\gamma_5, \tag{4.2.44}$$

which can be used in equation (4.2.43) to yields

$$\not{\nabla}^2 = D_\alpha D^\alpha + \frac{1}{2}\epsilon^{\alpha\beta}F_{\alpha\beta}(g_v\gamma_5 + g_a) - 2g_a\epsilon^{\alpha\beta}A_\alpha\partial_\beta + g_a(g_a - g_v\gamma_5)A_\alpha A^\alpha. \tag{4.2.45}$$

We then come back to equation (4.2.27), set $d = 2$, use equation (4.2.45) in it, apply the argument of function f to the positive exponential and scaling $k_\alpha \to \Lambda k_\alpha$ to get

$$\mathcal{A} = 2i \lim_{\Lambda\to\infty} \Lambda^2 \operatorname{tr} \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\gamma_5 f(k^2 + w), \tag{4.2.46}$$

where

$$\begin{aligned} w = &\frac{2i}{\Lambda}k_\alpha D^\alpha + \frac{1}{\Lambda^2}D_\alpha D^\alpha + \frac{1}{2\Lambda^2}\epsilon^{\alpha\beta}F_{\alpha\beta}(g_v\gamma_5 + g_a) \\ &- 2g_a\epsilon^{\alpha\beta}A_\alpha\left(\frac{i}{\Lambda}k_\beta + \frac{1}{\Lambda^2}\partial_\beta\right) + \frac{1}{\Lambda^2}g_a(g_a - g_v\gamma_5)A_\alpha A^\alpha. \end{aligned} \tag{4.2.47}$$

Now it is just a matter of to consider the Taylor expansion of the function f around the value $x = k^2$ and stop at the order $\mathcal{O}(\Lambda^{-2})$, since higher order terms will vanish in the limit $\Lambda \to \infty$. By making this we get

$$\mathcal{A} = 2i \lim_{\Lambda\to\infty} \Lambda^2 \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\left[f_0 + f_1 + f_2 + \mathcal{O}\big(\Lambda^{-3}\big)\right], \tag{4.2.48}$$

where

$$f_0 = f(k^2)(\operatorname{tr}\gamma_5), \tag{4.2.49a}$$

$$\begin{aligned} f_1 = f'(k^2)\Bigg\{&\frac{2i}{\Lambda}(\operatorname{tr}\gamma_5)k_\alpha D^\alpha + \frac{1}{\Lambda^2}(\operatorname{tr}\gamma_5)D_\alpha D^\alpha + \frac{1}{2\Lambda^2}\epsilon^{\alpha\beta}F_{\alpha\beta}[g_v(\operatorname{tr}1) + g_a(\operatorname{tr}\gamma_5)] \\ &-2g_a(\operatorname{tr}\gamma_5)\epsilon^{\alpha\beta}A_\alpha\left(\frac{i}{\Lambda}k_\beta + \frac{1}{\Lambda^2}\partial_\beta\right) + \frac{1}{\Lambda^2}g_a[g_a(\operatorname{tr}\gamma_5) - g_v(\operatorname{tr}1)]A_\alpha A^\alpha\Bigg\}, \end{aligned} \tag{4.2.49b}$$

$$\begin{aligned} f_2 = \frac{1}{2}f''(k^2)(\operatorname{tr}\gamma_5)\Bigg[&-\frac{4}{\Lambda^2}k_\alpha k_\beta D^\alpha D^\beta + \frac{4}{\Lambda^2}g_a\epsilon^{\alpha\beta}k_\sigma k_\beta D_\sigma A_\alpha \\ &+ \frac{4}{\Lambda^2}g_a\epsilon^{\alpha\beta}k_\sigma k_\beta A_\alpha D^\sigma - \frac{4}{\Lambda^2}g_a^2\epsilon^{\alpha\beta}\epsilon\lambda\nu k_\beta k_\nu A_\alpha A_\lambda + \Bigg] + \mathcal{O}\big(\Lambda^{-3}\big) \end{aligned} \tag{4.2.49c}$$

By using the equation (4.2.34) and that for (1+1) spacetime we have $\operatorname{tr}1 = 2$, the equations (4.2.49a) to (4.2.49c) become

$$f_0 = 0, \quad f_1 = \left(\frac{g_v}{\Lambda^2}\epsilon^{\alpha\beta}F_{\alpha\beta} - \frac{2g_v g_a}{\Lambda^2}A_\alpha A^\alpha\right)f'(k^2), \quad f_2 = \mathcal{O}\big(\Lambda^{-3}\big), \tag{4.2.50}$$

which can be used in the chiral anomaly (4.2.48) to give

$$\mathcal{A} = 2i\big(g_v\epsilon^{\alpha\beta}F_{\alpha\beta} - 2g_v g_a A_\alpha A^\alpha\big)\int\frac{\mathrm{d}^2k}{(2\pi)^2}f'(k^2), \tag{4.2.51}$$

in which the limit $\Lambda \to \infty$ was taken.

By using, then, the result (4.2.37) in the equation (4.2.51) we find that the chiral anomaly for the Chiral Schwinger model:

$$\mathcal{A} = -\frac{ig_v}{2\pi}\epsilon^{\alpha\beta}F_{\alpha\beta} + \frac{ig_v g_a}{\pi}A_\alpha A^\alpha. \tag{4.2.52}$$

It can be seen that when the vector coupling is taken $g_v \to e$ and the axial coupling is taken $g_a \to 0$ in chiral anomaly (4.2.52) for the Chiral Schwinger model, we recover the chiral anomaly for the usual Schwinger model given in equation (4.2.38).

It can be shown that the naive continuity equation for the axial-vector current $j_5^\alpha = \bar{\psi}\gamma^\alpha\gamma_5\psi$ in the Chiral Schwinger model is given just by

$$\partial_\alpha(\bar{\psi}\gamma^\alpha\gamma_5\psi) = 2im(\bar{\psi}\gamma_5\psi) + 2i\mu(\bar{\psi}\psi), \tag{4.2.53}$$

i.e., it receives contribution due to the fermionic axial mass.

Moreover, we use the (4.2.38) in the (4.2.28) to get the quantum continuity equation for the non-Hermitian QED (1+1):

$$\partial_\alpha(\bar{\psi}\gamma^\alpha\gamma_5\psi) = 2im(\bar{\psi}\gamma_5\psi) + 2i\mu(\bar{\psi}\psi) - 2i\Big(\frac{g_v}{4\pi}\epsilon^{\alpha\beta}F_{\alpha\beta}\Big) + 2i\Big(\frac{g_v g_a}{2\pi}A_\alpha A^\alpha\Big) \tag{4.2.54}$$

or for the Chiral Schwinger model, in which $\mu \to \pm m$,

$$\partial_\alpha(\bar{\psi}\gamma^\alpha\gamma_5\psi) = 2im\big[(\bar{\psi}\gamma_5\psi) \pm (\bar{\psi}\psi)\big] - 2i\Big(\frac{g_v}{4\pi}\epsilon^{\alpha\beta}F_{\alpha\beta}\Big) + 2i\Big(\frac{g_v g_a}{2\pi}A_\alpha A^\alpha\Big). \tag{4.2.55}$$

It can then be seen that the Chiral Schwinger model remains non chiral invariant after the quantization procedure, given that the right hand side of (4.2.55) is non null.

5 FINAL REMARKS

In this dissertation we have examined several aspects of the non-Hermitian QED. Our main result is its Pauli-Schrödinger Hamiltonian, elaborated in chapter 3. When one compares the Pauli-Schrödinger Hamiltonian of the non-Hermitian QED with that one for the usual QED it can be see two new main contributions: the spin coupled to the electric potential and the toroidal moment coupled to the magnetic field.

This Hamiltonian was obtained following two distinct ways: the non-relativistic limit applied to the Dirac equation and the non-relativistic limit applied at the tree-level QED.

In order to give a quick application of these results, we calculate the hydrogen energy spectrum for the Pauli-Schrödinger Hamiltonian of the non-Hermitian QED under two different conditions: spin on and spin off. The first case gives only a correction due the axial parameters μ and g_a of this model. The second case, on the other hand, give a complex spectrum due the presence of spin, which can be made real by suitable relations between the parameters of the model.

Another model of interest was the Chiral Schwinger model with non-unitary couplings, for which we calculated the vacuum polarization tensor, as can be seen in section 4.1. At first, it seems that the photon, as well as in the usual Schwinger model, acquires mass. However, for a better understand regarding the phenomenon of mass generation in the Chiral Schwinger model with non-unitary couplings it is necessary to perform a bozonization process of it, which is a task beyond the scope of this research.

The later result achieved was the chiral anomaly for the Chiral Schwinger model with non-unitary couplings, which was elaborated in section 4.2. This anomaly receives new contributions when compared with the usual Schwinger model giving, therefore, correction to the continuity equation for the axial-vector current.

The work at QED does not end with this dissertation. In order to provide some research possibilities, we elaborate three problems which will allow us to continue working in this topic.

1st: It is known that the Pauli-Schrödinger equation arises from the non-relativistic limit of the Dirac equation in an expansion to the order of m^{-1}, with m being the usual fermionic mass. However, in terms of the Foldy-Wouthuysen transformation of the usual Dirac equation it is possible to obtain, in the order of m^{-2}, corrections to the Pauli-Schrödinger equation, from which Darwin's famous term arises: $(e/8m^2)\boldsymbol{\nabla}\cdot\mathbf{E}$, with $\mathbf{E}$ being an external electromagnetic field (FOLDY; WOUTHUYSEN, 1950).

On the other hand, the Foldy-Wouthuysen transformation of the Dirac equation with axial mass is already known (ALEXANDRE; BENDER, 2015). The following question then arises: **what contributions does the axial mass provide to the Darwin term and the Pauli-Schrödinger equation, as a whole, up to the second order of expansion of the inverse of the ordinary mass?**

2nd: The vacuum polarization tensor is a quantity of extreme interest in any model, considering that it contributes to the propagator (PESKIN; SCHROEDER, 1995, p. 246), which, in turn, is used in calculating the amplitudes of the Feynman diagrams, and also gives corrections to the Coulombian interaction as known as Uehling potential (PESKIN; SCHROEDER, 1995, p. 255).

In the case of non-Hermitian QED, this tensor has already been reported (ALEXANDRE; BENDER; MILLINGTON, 2015), whose method of obtaining it follows a traditional calculation where the spinor right- and left-hand components are treated separately, thus avoiding dealing with the dimensional regularization of the axial matrix γ_5.

There is, however, another method that tackles the dimensional regularization of the axial matrix γ_5 (THOMPSON; YU, 1985). In this approach, the interaction vertex is expanded using free parameters, violating gauge invariance. After the regularization is realized in n-dimensions, the limit $n \to 4$ is taken. This method was implemented in dimensions (1+1) and used to calculate the vacuum polarization tensor of the chiral Schwinger model with unitary couplings (YU; YEUNG, 1987b), with non-unitary couplings [this dissertation] and in the axial model (BARCELOS-NETO; SOUZA, 1989). In dimensions (3+1), the same method was used to perturbatively calculate the chiral anomaly (YU; YEUNG, 1987b).

A problem, then, would be the following: **calculate the vacuum polarization tensor using dimensional regularization with interaction vertex extension for the non-Hermitian QED and compare the results with the already published result, in order to verify the veracity of the method.**

3rd: We know that temperature is an essential factor in physics and there are two ways to add it to field theory: the Matsubara formalism, where the temporal coordinate is deformed so as to be imaginary; and the real-time formalism, which allows analyzing the effects of temperature along the temporal evolution of the system. Within the real-time formalism, there are still two approaches: the closed-time path, which increases the degrees of freedom and modifies Green's functions; and Termofield Dynamics (TFD), where a thermal vacuum state is constructed (KHANNA et al., 2009).

With TFD it is possible to calculate the Breit-Wheeler process at finite temperature (CABRAL; SANTOS; BUFALO, 2023), the Compton scattering (CABRAL; SANTOS, 2023) and the Casimir effect (PRATA; SANTOS; KHANNA, 2023).

Based on this, the following question arises: **what contribution does non-Hermitian deformation provide to the aforementioned finite temperature processes?**

REFERENCES

AAD, G. et al. (Atlas Collaboration). Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC. **Physics Letters B**, Elsevier B.V., v. 716, n. 1, p. 1–29, Sept. 2012. Available from: <https://www.sciencedirect.com/science/article/pii/S037026931200857X>. Visited on: 16 Sept. 2024.

ABI, B. et al. (Muon $g-2$ Collaboration). Measurement of the Positive Muon Anomalous Magnetic Moment to 0.46 ppm. **Physical Review Letters**, American Physical Society, v. 126, n. 14, p. 141801, Apr. 2021. Available from: <https://link.aps.org/doi/10.1103/PhysRevLett.126.141801>. Visited on: 16 Sept. 2024.

AGUILLARD, D. P. et al. (Muon $g-2$ Collaboration). Measurement of the Positive Muon Anomalous Magnetic Moment to 0.20 ppm. **Physical Review Letters**, American Physical Society, v. 131, n. 16, p. 161802, Oct. 2023. Available from: <https://link.aps.org/doi/10.1103/PhysRevLett.131.161802>. Visited on: 18 Sept. 2024.

ALEXANDRE, J.; BENDER, C. M. Foldy–Wouthuysen transformation for non-Hermitian Hamiltonians. **Journal of Physics A: Mathematical and Theoretical**, IOP Publishing, v. 48, n. 18, p. 185403, Apr. 2015. Available from: <https://dx.doi.org/10.1088/1751-8113/48/18/185403>. Visited on: 16 Sept. 2024.

ALEXANDRE, J.; BENDER, C. M.; MILLINGTON, P. Light neutrino masses from a non-Hermitian Yukawa theory. **Journal of Physics: Conference Series**, IOP Publishing, v. 873, n. 1, p. 012047, July 2017. Available from: <https://dx.doi.org/10.1088/1742-6596/873/1/012047>. Visited on: 16 Sept. 2024.

ALEXANDRE, J.; BENDER, C. M.; MILLINGTON, P. Non-Hermitian extension of gauge theories and implications for neutrino physics. **Journal of High Energy Physics**, Springer Nature, v. 2015, n. 11, p. 111, Nov. 2015. Available from: <https://doi.org/10.1007/JHEP11(2015)111>. Visited on: 16 Sept. 2024.

ALEXANDRE, J.; ELLIS, J.; MILLINGTON, P. Discrete spacetime symmetries and particle mixing in non-Hermitian scalar quantum field theories. **Physical Review D**, American Physical Society (APS), v. 102, n. 12, p. 125030, Dec. 2020. Available from: <https://link.aps.org/doi/10.1103/PhysRevD.102.125030>. Visited on: 16 Sept. 2024.

ALEXANDRE, J.; ELLIS, J.; MILLINGTON, P.; SEYNAEVE, D. Spontaneous symmetry breaking and the Goldstone theorem in non-Hermitian field theories. **Physical Review D**, American Physical Society, v. 98, n. 4, p. 045001, Aug. 2018. Available from: <https://link.aps.org/doi/10.1103/PhysRevD.98.045001>. Visited on: 16 Sept. 2024.

ALEXANDRE, J.; MILLINGTON, P.; SEYNAEVE, D. Symmetries and conservation laws in non-Hermitian field theories. **Physical Review D**, American Physical Society, v. 96, n. 6, p. 065027, Sept. 2017. Available from: <https://link.aps.org/doi/10.1103/PhysRevD.96.065027>. Visited on: 16 Sept. 2024.

AOYAMA, T.; KINOSHITA, T.; NIO, M. Theory of the Anomalous Magnetic Moment of the Electron. **Atoms**, MDPI, v. 7, n. 1, p. 28, Feb. 2019. Available from: <https://doi.org/10.3390/atoms7010028>. Visited on: 16 Sept. 2024.

ASHIDA, Y.; FURUKAWA, S.; UEDA, M. Parity-time-symmetric quantum critical phenomena. **Nature Communications**, Springer Science and Business Media LLC, v. 8, n. 1, p. 15791, June 2017. Available from: <https://doi.org/10.1038/ncomms15791>. Visited on: 16 Sept. 2024.

ASHIDA, Y.; GONG, Z.; UEDA, M. Non-Hermitian physics. **Advances in Physics**, Taylor & Francis, v. 69, n. 3, p. 249–435, July 2020. Available from: <https://doi.org/10.1080/00018732.2021.1876991>. Visited on: 16 Sept. 2024.

BAIKOV, P. A.; IL'IN, V. A. Status of γ_5 in dimensional regularization. **Theoretical and Mathematical Physics**, Springer Nature, v. 88, n. 2, p. 789, Aug. 1991. Available from: <https://doi.org/10.1007/BF01019107>. Visited on: 16 Sept. 2024.

BARCELOS-NETO, J.; SOUZA, C. F. Dimensional regularisation in the axial model. **Journal of Physics G: Nuclear and Particle Physics**, IOP Publishing, v. 15, n. 5, p. 555, May 1989. Available from: <https://dx.doi.org/10.1088/0954-3899/15/5/009>. Visited on: 16 Sept. 2024.

BARON, J. et al. Dimensional regularisation in the axial model. **Science**, AAAS, v. 343, n. 6168, p. 269, May 1989. Available from: <https://www.science.org/doi/abs/10.1126/science.1248213>. Visited on: 18 Sept. 2024.

BENDER, C. M.; BERNTSON, B. K., et al. Observation of PT phase transition in a simple mechanical system. **American Journal of Physics**, AIP Publishing, v. 81, n. 3, p. 173, Mar. 2013. Available from: <https://doi.org/10.1119/1.4789549>. Visited on: 16 Sept. 2024.

BENDER, C. M.; BOETTCHER, S. Real Spectra in Non-Hermitian Hamiltonians Having $\mathscr{P}\mathscr{T}$ Symmetry. **Physical Review Letters**, American Physical Society, v. 80, n. 24, p. 5243, June 1998. Available from: <https://link.aps.org/doi/10.1103/PhysRevLett.80.5243>. Visited on: 16 Sept. 2024.

BENDER, C. M.; BRODY, D. C., et al. Faster than Hermitian Quantum Mechanics. **Physical Review Letters**, American Physical Society, v. 98, n. 4, p. 040403, Jan. 2007. Available from: <https://link.aps.org/doi/10.1103/PhysRevLett.98.040403>. Visited on: 16 Sept. 2024.

BENDER, C. M.; JONES, H. F.; RIVERS, R. J. Dual $\mathscr{P}\mathscr{T}$-symmetric quantum field theories. **Physics Letters B**, Elsevier B.V., v. 625, 3–4, p. 333, Oct. 2005. Available from: <https://doi.org/10.1016/j.physletb.2005.08.087>. Visited on: 16 Sept. 2024.

BENDER, C. M.; MILTON, K. A. Nonperturbative calculation of symmetry breaking in quantum field theory. **Physical Review D**, American Physical Society, v. 55, n. 6, r3255, Mar. 1997. Available from: <https://link.aps.org/doi/10.1103/PhysRevD.55.R3255>. Visited on: 16 Sept. 2024.

BENDER, C. M. **PT Symmetry in Quantum and Classical Physics**. Singapore: World Scientific Publishing Europe, 2019.

BERTLMANN, R. A. **Anomalies in Quantum Field Theory**. Oxford: Clarendon Press, 1996. (International Series of Monographs on Physics, 91).

BEYGI, A. **Non-Hemitian PT-Symmetric Theories**. 2019. Dissertação – Heidelberg University, Heidelberg. Available from: <http://www.ub.uni-heidelberg.de/archiv/26903>. Visited on: 16 Sept. 2024.

BILENKY, S. **Introduction to the Physics of Massive and Mixed Neutrinos**. 2. ed. Cham: Springer International Publishing AG, 2018. (Lecture Notes in Physics, 947).

BOUDJEMA, F. et al. Electromagnetic Properties of Generalized Majorana Particles. **Physical Review Letters**, American Physical Society, v. 62, n. 8, p. 852, Feb. 1989. Available from: <https://link.aps.org/doi/10.1103/PhysRevLett.62.852>. Visited on: 16 Sept. 2024.

BUKINA, E. N.; DUBOVIK, V. M.; KUZNETSOV, V. E. Transition radiation of the neutrino toroid dipole moment. **Physics Letters B**, Elsevier B.V., v. 435, n. 1–2, p. 134, Sept. 1998. Available from: <https://doi.org/10.1016/S0370-2693(98)00776-X>. Visited on: 16 Sept. 2024.

CABRAL, D. S.; SANTOS, A. F. Compton scattering in TFD formalism. **The European Physical Journal C**, Springer Nature, v. 83, n. 1, p. 25, Jan. 2023. Available from: <https://doi.org/10.1140/epjc/s10052-023-11182-x>. Visited on: 16 Sept. 2024.

CABRAL, D. S.; SANTOS, A. F.; BUFALO, R. Thermal pair production from photon-photon collision: Breit–Wheeler process at finite temperature. **The European Physical Journal C**, Springer Nature, v. 83, n. 12, p. 1113, Dec. 2023. Available from: <https://doi.org/10.1140/epjc/s10052-023-12281-5>. Visited on: 16 Sept. 2024.

CABRAL-ROSETTI, L. G.; MONDRAGÓN, M.; PÉREZ, E. R. Toroidal Dipole Moment of a Massless Neutrino. **AIP Conference Proceedings**, AIP Publishing, v. 1116, n. 1, p. 447, Apr. 2009. Available from: <https://doi.org/10.1063/1.3131597>. Visited on: 16 Sept. 2024.

CABRAL-ROSETTI, L. G.; MORENO, M.; ROSADO, A. Dirac Neutrino Anapole Moment. **AIP Conference Proceedings**, AIP Publishing, v. 623, n. 1, p. 347, July 2002. Available from: <https://doi.org/10.1063/1.1489777>. Visited on: 16 Sept. 2024.

CHANOWITZ, M.; FURMAN, M.; HINCHLIFFE, I. The axial current in dimensional regularization. **Nuclear Physics B**, Elsevier B.V., v. 159, n. 1–2, p. 225, Nov. 1979. Available from: <https://doi.org/10.1016/0550-3213(79)90333-X>. Visited on: 16 Sept. 2024.

CHATRCHYAN, S. et al. (CMS Collaboration). Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC. **Physics Letters B**, Elsevier B.V., v. 716, n. 1, p. 30, Sept. 2012. Available from: <https://doi.org/10.1016/j.physletb.2012.08.021>. Visited on: 16 Sept. 2024.

DAS, A. **Lectures on Quantum Field Theory**. 2. ed. Singapore: World Scientific, 2021.

DONOGHUE, J. F. General relativity as an effective field theory: The leading quantum corrections. **Physical Review D**, American Physical Society, v. 50, n. 6, p. 3874, Sept. 1994. Available from: <https://link.aps.org/doi/10.1103/PhysRevD.50.3874>. Visited on: 17 Sept. 2024.

DUBOVIK, V. M.; TUGUSHEV, V. V. Toroid moments in electrodynamics and solid-state physics. **Physics Reports**, Elsevier B.V., v. 187, n. 4, p. 145, Mar. 1990. Available from: <https://doi.org/10.1016/0370-1573(90)90042-Z>. Visited on: 17 Sept. 2024.

DYSON, F. J. Energy in the universe. **Scientific American**, JSTOR, v. 225, n. 3, p. 50, Sept. 1971. Available from: <https://www.jstor.org/stable/24923116>. Visited on: 17 Sept. 2024.

EDERER, C.; SPALDIN, N. A. Towards a microscopic theory of toroidal moments in bulk periodic crystals. **Physical Review B**, American Physical Society, v. 76, n. 21, p. 214404, Dec. 2007. Available from: <https://link.aps.org/doi/10.1103/PhysRevB.76.214404>. Visited on: 17 Sept. 2024.

EISBERG, R. M. **Fundamentals of modern physics**. New York: John Wiley & Sons, 1961.

FAN, X. et al. Measurement of the Electron Magnetic Moment. **Physical Review Letters**, American Physical Society, v. 130, n. 7, p. 071801, Feb. 2023. Available from: <https://link.aps.org/doi/10.1103/PhysRevLett.130.071801>. Visited on: 18 Sept. 2024.

FOLDY, L. L.; WOUTHUYSEN, S. A. On the Dirac Theory of Spin 1/2 Particles and Its Non-Relativistic Limit. **Physical Review**, American Physical Society, v. 78, n. 1, p. 29, Apr. 1950. Available from: <https://link.aps.org/doi/10.1103/PhysRev.78.29>. Visited on: 17 Sept. 2024.

FUJIKAWA, K.; SUZUKI, H. **Path Integrals and Quantum Anomalies**. Oxford: Clarendon Press, 2004. (International Series of Monographs on Physics, 122).

GOLDBERG, D. **The Standard Model in a Nutshell**. Princeton: Princeton University Press, 2017.

GRAEFE, E.-M. et al. Classical and quantum dynamics in the (non-Hermitian) Swanson oscillator. **Journal of Physics A: Mathematical and Theoretical**, IOP Publishing, v. 48, n. 5, p. 055301, Jan. 2015. Available from: <https://dx.doi.org/10.1088/1751-8113/48/5/055301>. Visited on: 17 Sept. 2024.

GRIFFITHS, D. J. **Introduction to elementary particles**. 2. ed. Weinheim: Wiley-VCH, 2008.

HATANO, N.; NELSON, D. R. Localization Transitions in Non-Hermitian Quantum Mechanics. **Physical Review Letters**, American Physical Society, v. 77, n. 3, p. 570, July 1996. Available from: <https://link.aps.org/doi/10.1103/PhysRevLett.77.570>. Visited on: 17 Sept. 2024.

HATANO, N.; NELSON, D. R. Vortex pinning and non-Hermitian quantum mechanics. **Physical Review B**, American Physical Society, v. 56, n. 14, p. 8651, Oct. 1997. Available from: <https://link.aps.org/doi/10.1103/PhysRevB.56.8651>. Visited on: 17 Sept. 2024.

JACKIW, R.; RAJARAMAN, R. Vector-Meson Mass Generation by Chiral Anomalies. **Physical Review Letters**, American Physical Society, v. 54, n. 12, p. 1219, Mar. 1985. Available from: <https://link.aps.org/doi/10.1103/PhysRevLett.54.1219>. Visited on: 17 Sept. 2024.

JEGERLEHNER, F. **The Anomalous Magnetic Moment of the Muon**. 2. ed. Cham: Springer International Publishing AG, 2017. (Springer Tracts in Modern Physics, 274).

JU, C.-Y. et al. Non-Hermitian Hamiltonians and no-go theorems in quantum information. **Physical Review A**, American Physical Society, v. 100, n. 6, p. 062118, Dec. 2019. Available from: <https://link.aps.org/doi/10.1103/PhysRevA.100.062118>. Visited on: 17 Sept. 2024.

KAYSER, B. Majorana neutrinos and their electromagnetic properties. **Physical Review D**, American Physical Society, v. 26, n. 7, p. 1662, Oct. 1982. Available from: <https://link.aps.org/doi/10.1103/PhysRevD.26.1662>. Visited on: 17 Sept. 2024.

KHANNA, F. C. et al. (Eds.). **Thermal quantum field theory**: Algebraic Aspects and Applications. Singapore: World Scientific, 2009.

KÖRNER, J. G.; KREIMER, D.; SCHILCHER, K. A practicable γ_5-scheme in dimensional regularization. **Zeitschrift für Physik C Particles and Fields**, Springer Nature, v. 54, n. 3, p. 503, Sept. 1992. Available from: <https://doi.org/10.1007/BF01559471>. Visited on: 17 Sept. 2024.

KRAMIDA, A.; RALCHENKO, Y.; READER, J. (NIST ASD Team). 2023. Available from: <https://doi.org/10.18434/T4W30F>. Visited on: 18 Sept. 2024.

KRASNOK, A.; ALÙ, A. Low-Symmetry Nanophotonics. **ACS Photonics**, American Chemical Society, v. 9, n. 1, p. 2, Jan. 2022. Available from: <https://pubs.acs.org/doi/abs/10.1021/acsphotonics.1c00968>. Visited on: 17 Sept. 2024.

KRISHNAMOORTHY, H. N. S. et al. Topological Insulator Metamaterials. **Chemical Reviews**, American Chemical Society, v. 123, n. 8, p. 4416, Mar. 2023. Available from: <https://doi.org/10.1021/acs.chemrev.2c00594>. Visited on: 17 Sept. 2024.

LIN, Z. et al. Experimental quantum simulation of non-Hermitian dynamical topological states using stochastic Schrödinger equation. **npj Quantum Information**, Springer Nature, v. 8, n. 1, p. 77, June 2022. Available from: <https://doi.org/10.1038/s41534-022-00587-3>. Visited on: 13 Sept. 2023.

MAKRIS, K. G. et al. Beam Dynamics in $\mathscr{PT}$ Symmetric Optical Lattices. **Physical Review Letters**, American Physical Society, v. 100, n. 10, p. 103904, Mar. 2008. Available from: <https://link.aps.org/doi/10.1103/PhysRevLett.100.103904>. Visited on: 18 Sept. 2024.

MATSUMOTO, N. et al. Continuous Phase Transition without Gap Closing in Non-Hermitian Quantum Many-Body Systems. **Physical Review Letters**, American Physical Society, v. 125, n. 26, p. 260601, Dec. 2020. Available from: <https://link.aps.org/doi/10.1103/PhysRevLett.125.260601>. Visited on: 18 Sept. 2024.

MIDYA, B.; ZHAO, H.; FENG, L. Non-Hermitian photonics promises exceptional topology of light. **Nature Communications**, Springer Science and Business Media LLC, v. 9, n. 1, July 2018. ISSN 2041-1723.

MILLINGTON, P. Non-Hermiticity: a new paradigm for model building in particle physics. In: PROCEEDINGS of The European Physical Society Conference on High Energy Physics — PoS(EPS-HEP2021). Sissa Medialab, Feb. 2022. v. 398. (EPS-HEP2021), p. 735. Available from: <https://doi.org/10.22323/1.398.0735>. Visited on: 18 Sept. 2024.

MOSTAFAZADEH, A. Pseudo-Hermiticity versus PT symmetry: The necessary condition for the reality of the spectrum of a non-Hermitian Hamiltonian. **Journal of Mathematical Physics**, AIP Publishing, v. 43, n. 1, p. 205, Jan. 2002a. Available from: <https://doi.org/10.1063/1.1418246>. Visited on: 18 Sept. 2024.

MOSTAFAZADEH, A. Pseudo-Hermiticity versus PT-symmetry II. A complete characterization of non-Hermitian Hamiltonians with a real spectrum. **Journal of Mathematical Physics**, AIP Publishing, v. 43, n. 5, p. 2814, May 2002b. Available from: <https://doi.org/10.1063/1.1461427>. Visited on: 18 Sept. 2024.

MOSTAFAZADEH, A. Pseudo-Hermiticity versus PT-symmetry III: Equivalence of pseudo-Hermiticity and the presence of antilinear symmetries. **Journal of Mathematical Physics**, AIP Publishing, v. 43, n. 8, p. 3944, Aug. 2002c. Available from: <https://doi.org/10.1063/1.1489072>. Visited on: 18 Sept. 2024.

NELSON, D. R.; SHNERB, N. M. Non-Hermitian localization and population biology. **Physical Review E**, American Physical Society, v. 58, n. 2, p. 1383, Aug. 1998. Available from: <https://link.aps.org/doi/10.1103/PhysRevE.58.1383>. Visited on: 18 Sept. 2024.

NOVOTNÝ, J. Axial anomaly and dimensional regularization (A review). **Czechoslovak Journal of Physics**, Springer Nature, v. 44, n. 7, p. 633, July 1994. Available from: <https://doi.org/10.1007/BF01694837>. Visited on: 18 Sept. 2024.

OHLSSON, T. Non-Hermitian neutrino oscillations in matter with PT symmetric Hamiltonians. **Europhysics Letters**, IOP Publishing, v. 113, n. 6, p. 61001, Apr. 2016. Available from: <https://doi.org/10.1209/0295-5075/113/61001>. Visited on: 18 Sept. 2024.

OHLSSON, T.; ZHOU, S. Transition probabilities in the two-level quantum system with PT-symmetric non-Hermitian Hamiltonians. **Journal of Mathematical Physics**, AIP Publishing, v. 61, n. 5, p. 052104, May 2020. Available from: <https://doi.org/10.1063/5.0002958>. Visited on: 18 Sept. 2024.

PEIERLS, R. E. Complex Eigenvalues in Scattering Theory. **Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences**, The Royal Society, v. 253, n. 1272, p. 16, Nov. 1959. Available from: <http://www.jstor.org/stable/100905>. Visited on: 18 Sept. 2024.

PESKIN, M. E.; SCHROEDER, D. V. **An Introduction to Quantum Field Theory**. Reading: Addison-Wesley Publishing Company, 1995. (The advanced book program).

PRATA, V. G.; SANTOS, A. F.; KHANNA, F. C. Stefan–Boltzmann law and Casimir effect for dark photons. **The European Physical Journal C**, Springer Nature, v. 83, n. 11, p. 1090, Nov. 2023. Available from: <https://doi.org/10.1140/epjc/s10052-023-12262-8>. Visited on: 18 Sept. 2024.

REES, M. Numerical Coincidences and ‘Tuning’ in Cosmology. **Astrophysics and Space Science**, Springer Nature, v. 285, n. 2, p. 375, July 2003. Available from: <https://doi.org/10.1023/A:1025444627554>. Visited on: 18 Sept. 2024.

RYANG, S. Bosonization in the chiral Schwinger model. **Physical Review D**, American Physical Society, v. 35, n. 10, p. 3158, May 1987. Available from: <https://link.aps.org/doi/10.1103/PhysRevD.35.3158>. Visited on: 18 Sept. 2023.

SCHWARTZ, M. D. **Quantum Field Theory and the Standard Model**. Cambridge: Cambridge University Press, 2014.

SPALDIN, N. A.; FIEBIG, M.; MOSTOVOY, M. The toroidal moment in condensed-matter physics and its relation to the magnetoelectric effect. **Journal of Physics: Condensed Matter**, IOP Publishing, v. 20, n. 43, p. 434203, Oct. 2008. Available from: <https://dx.doi.org/10.1088/0953-8984/20/43/434203>. Visited on: 18 Sept. 2024.

SUEKANE, F. **Neutrino Oscillations**: A Practical Guide to Basics and Applications. Tokyo: Springer Japan, 2015. (Lecture Notes in Physics, 898).

SWANSON, M. S. Transition elements for a non-Hermitian quadratic Hamiltonian. **Journal of Mathematical Physics**, AIP Publishing, v. 45, n. 2, p. 585, Feb. 2004. Available from: <https://doi.org/10.1063/1.1640796>. Visited on: 18 Sept. 2024.

TEGMARK, M. On the dimensionality of spacetime. **Classical and Quantum Gravity**, IOP Publishing, v. 14, n. 4, l69, Apr. 1997. Available from: <https://dx.doi.org/10.1088/0264-9381/14/4/002>. Visited on: 18 Sept. 2024.

THOMPSON, G.; YU, H.-L. γ_5 in dimensional regularization. **Physics Letters B**, Elsevier B.V., v. 151, n. 2, p. 119, Feb. 1985. Available from: <https://www.sciencedirect.com/science/article/pii/0370269385913978>. Visited on: 18 Sept. 2024.

WEINBERG, S. Anthropic Bound on the Cosmological Constant. **Physical Review Letters**, American Physical Society, v. 59, n. 22, p. 2607, Nov. 1987. Available from: <https://link.aps.org/doi/10.1103/PhysRevLett.59.2607>. Visited on: 18 Sept. 2024.

WILCZEK, F. The Universe is a Strange Place. **Nuclear Physics B - Proceedings Supplements**, Elsevier BV, v. 134, p. 3, Sept. 2004. Proceedings of the Second International Conference on Particle and Fundamental Physics in Space (SpacePart '03). Available from: <https://www.sciencedirect.com/science/article/pii/S0920563204002919>. Visited on: 18 Sept. 2024.

YU, H.-L.; YEUNG, W. B. Dimensional regularization and perturbative solution of the chiral Schwinger model. **Physical Review D**, American Physical Society, v. 35, n. 2, p. 636, Jan. 1987a. Available from: <https://link.aps.org/doi/10.1103/PhysRevD.35.636>. Visited on: 18 Sept. 2024.

YU, H.-L.; YEUNG, W. B. Dimensional regularization and the γ_5 axial anomaly. **Physical Review D**, American Physical Society, v. 35, n. 12, p. 3955, June 1987b. Available from: <https://link.aps.org/doi/10.1103/PhysRevD.35.3955>. Visited on: 18 Sept. 2024.